

## DUALIZABILITY AND INDEX OF SUBFACTORS

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ABSTRACT. In this paper, we develop the theory of bimodules over von Neumann algebras, with an emphasis on categorical aspects. We clarify the relationship between dualizability and finite index. We also show that, for von Neumann algebras with finite-dimensional centers, the Haagerup  $L^2$ -space and Connes fusion are functorial with respect to homomorphisms of finite index. Along the way, we describe a string diagram notation for maps between bimodules that are not necessarily bilinear.

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## 1. INTRODUCTION

The operation  $({}_A H_B, {}_B K_C) \mapsto {}_A H \boxtimes_B K_C$  of Connes fusion is an associative product on bimodules between von Neumann algebras [3, 26, 31]. It behaves formally like a tensor product, but its construction is somewhat involved and relies heavily on the notion of non-commutative  $L^2$ -space [4, 11, 32]. Connes fusion is designed so as to have the  $L^2$ -space as its identity:  ${}_A L^2 A \boxtimes_A H_B \cong {}_A H_B \cong {}_A H \boxtimes_B {}_B L^2 B$ . Altogether, von Neumann algebras, their bimodules, and bimodule intertwiners form a symmetric monoidal bicategory. As in any bicategory, one can talk about a morphism being dualizable<sup>1</sup> [21, 30]: a bimodule  ${}_A H_B$  is called dualizable, with dual  ${}_B \bar{H}_A$ , if it comes equipped with maps

$$(1.1) \quad R^* : {}_A H \boxtimes_B \bar{H}_A \rightarrow {}_A L^2(A)_A \quad S : {}_B L^2(B)_B \rightarrow {}_B \bar{H} \boxtimes_A H_B$$

subject to the duality equations  $(R^* \otimes 1)(1 \otimes S) = 1$ ,  $(1 \otimes R^*)(S \otimes 1) = 1$ . The dual bimodule  ${}_B \bar{H}_A$  is well defined up to unique isomorphism. In fact, under suitable normalization conditions on the duality maps  $R^*$  and  $S$ , the dual bimodule is well defined up to unique *unitary* isomorphism. If  $A$  and  $B$  are factors one can then define the *statistical dimension* of  ${}_A H_B$  as  $R^* R = S^* S$  [20].

A subfactor  $N \subset M$  has an invariant called the minimal index  $[M : N] \in \mathbb{R}_{\geq 1} \cup \{\infty\}$  [12, 14, 16], and this index is finite if and only if the bimodule  ${}_N L^2 M_M$  is dualizable. When that bimodule is dualizable, the minimal index may be defined

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<sup>1</sup>As written, equation (1.1) corresponds to the notion of *left* dualizability, but since our bicategory has a  $*$ -structure, there is no difference between left and right dualizability.

as the square of the statistical dimension of  ${}_N L^2 M_M$ . We show that this definition agrees with the traditional notion of minimal index, by comparing the squared statistical dimension with the optimal bound of a Pimsner–Popa inequality for the subfactor [16, 24].

Given two von Neumann algebras  $A$  and  $B$  that have finite-dimensional centers (in other words are finite direct sums of factors), we call a homomorphism  $f : A \rightarrow B$  *finite* if the bimodule  ${}_A L^2 B_B$  is dualizable. Restricting attention to these finite homomorphisms makes the  $L^2$  construction functorial:

**Theorem.** *The assignment*

$$A \mapsto L^2(A)$$

*is a functor from the category*

$$\left\{ \begin{array}{ll} \text{objects:} & \text{von Neumann algebras with finite-dimensional center} \\ \text{morphisms:} & \text{finite homomorphisms} \end{array} \right.$$

*to the category of Hilbert spaces and bounded linear maps.*

We conjecture that this functor in fact extends to the category of all von Neumann algebras and finite homomorphisms.

The Connes fusion  $H \boxtimes_A K$  is certainly functorial in  $H$  and  $K$ . We show that it is moreover simultaneously functorial in the three variables  $H$ ,  $K$  and  $A$ :

**Theorem.** *The assignment*

$$(H, A, K) \mapsto H \boxtimes_A K$$

*is a functor from the category*

$$\left\{ \begin{array}{ll} \text{objects:} & \text{triples } (H, A, K) \text{ where } A \text{ is a von Neumann algebra} \\ & \text{with finite-dimensional center, } H \text{ is a right } A\text{-module,} \\ & \text{and } K \text{ is a left } A\text{-module} \\ \text{morphisms:} & \text{triples } (h, \alpha, k) \text{ where } \alpha \text{ is a finite homomorphism} \\ & A_1 \rightarrow A_2, h \text{ is a module map } H_1 \rightarrow H_2, \text{ and } k \text{ is} \\ & \text{a module map } K_1 \rightarrow K_2 \end{array} \right.$$

*to the category of Hilbert spaces and bounded linear maps.*

Note that our techniques and results all apply equally well to type *I*, *II*, and *III* von Neumann algebras.

**Outline.** Our new graphical notation is described in section 2, along with preliminaries concerning von Neumann algebras and Haagerup’s  $L^2$ -space. We emphasize the fact that it is not necessary to choose a state  $\phi : A \rightarrow \mathbb{C}$  in order to define  $L^2(A)$  [11]. In section 3, we discuss Connes fusion and some of its elementary properties. In section 4, we investigate the concept of dualizable bimodules. We prove that the endomorphism algebra  $\text{End}({}_A H_B)$  of a dualizable bimodule is finite-dimensional and is equipped with a canonical trace. Moreover, we show the dual is well defined up to unique unitary isomorphism. In section 5, we define the statistical dimension of a dualizable bimodule and introduce the categorical definition of the minimal index of a subfactor, namely  $[M : N] := \dim({}_N L^2 M_M)^2$ . In section 6, we present our new results: the functoriality of  $L^2$  and of Connes fusion. Finally, in section 7, we use the Pimsner–Popa inequality to show that the categorical definition of the minimal index agrees with other definitions [12, 13, 16, 24]. We end the paper by presenting a collection of useful inequalities for the minimal index.

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## 2. PRELIMINARIES

**Von Neumann algebras.** Given a complex Hilbert space  $H$ , let  $\mathbf{B}(H)$  denote its algebra of bounded operators. The ultraweak topology on  $\mathbf{B}(H)$  is the topology of pointwise convergence with respect to the pairing with its predual, the trace class operators.

**Definition 2.1.** A von Neumann algebra is a topological  $*$ -algebra that is embeddable as a closed subalgebra of  $\mathbf{B}(H)$  with respect to the ultraweak topology.

**Definition 2.2.** Let  $A$  be a von Neumann algebra. A left (right)  $A$ -module is a Hilbert space  $H$  equipped with a continuous homomorphism from  $A$  (respectively  $A^{op}$ ) to  $\mathbf{B}(H)$ . We will use the notation  ${}_AH$  (respectively  $H_A$ ) to denote the fact that  $H$  is a left (right)  $A$ -module.

The main distinguishing feature of the representation theory of von Neumann algebras is the following:

**Proposition 2.3** ([6, Remark 2.1.3. (iii)]). *Let  $A$  be a von Neumann algebra and let  $H$  and  $K$  be two faithful left  $A$ -modules. Then  $H \otimes \ell^2 \cong K \otimes \ell^2$ . In particular, any  $A$ -module is isomorphic to a direct summand of  $H \otimes \ell^2$ .*  $\square$

If the Hilbert spaces  $H$  and  $K$  in this proposition are separable, then  $\ell^2$  can be taken to mean  $\ell^2(\mathbb{N})$ . Otherwise, the proposition is true for  $\ell^2 = \ell^2(X)$ , for  $X$  some set of sufficiently large cardinality.

The spatial tensor product  $A_1 \bar{\otimes} A_2$  of von Neumann algebras  $A_i \subset \mathbf{B}(H_i)$  is the closure in  $\mathbf{B}(H_1 \otimes H_2)$  of the algebraic tensor product  $A_1 \otimes_{alg} A_2$ ; by the above proposition, this closure is independent of the choices of Hilbert spaces  $H_1$  and  $H_2$ . The spatial tensor product is a symmetric monoidal structure on the category of von Neumann algebras.

**The Haagerup  $L^2$ -space.** Given a von Neumann algebra  $A$ , the space of continuous linear functionals  $A \rightarrow \mathbb{C}$  forms a Banach space  $A_* = L^1(A)$  called the predual of  $A$ . It is equipped with two commuting  $A$  actions given by  $(a\phi b)(x) := \phi(bxa)$  and a cone  $L_+^1(A) := \{\phi \in A_* \mid \phi(x) \geq 0 \ \forall x \in A_+\}$ . Here,  $A_+ := \{a^*a \mid a \in A\}$  is the set of positive elements of  $A$ .

The Haagerup  $L^2$ -space of  $A$  is an  $A$ - $A$ -bimodule that is canonically associated to  $A$ . It is denoted  $L^2(A)$ , and its construction does not depend on any choices [11]. It is the completion of

$$\bigoplus_{\phi \in L_+^1(A)} \mathbb{C}\sqrt{\phi}$$

with respect to some pre-inner product. We will provide more details of the construction of  $L^2(A)$  at the beginning of section 6.

*Remark 2.4.* At this point,  $\sqrt{\phi} \in L^2A$  should be treated as a formal symbol. However, there exists a natural  $*$ -algebra structure on  $\bigoplus_p L^pA$  in which  $\sqrt{\phi}$  is the (unique positive) square root of  $\phi \in L^1A$  — see Remark 6.3 for further details.

*Remark 2.5.* There is an isomorphism  $L^2(A) \cong L^2(A^{op})$  under which the left action of  $A$  on  $L^2A$  is equal to the right action of  $A^{op}$  on  $L^2(A^{op})$ , and the right action of  $A$  on  $L^2A$  is equal to the left action of  $A^{op}$  on  $L^2(A^{op})$ .

The  $L^2$  construction is compatible with direct sums, in the sense that  $L^2(A \oplus B) = L^2(A) \oplus L^2(B)$ . This is a corollary of the relationship expressed in the following lemma, between the  $L^2$ -space construction and the operation of taking the corner algebra  $pAp$  associated to a projection  $p \in A$ .

**Lemma 2.6** ([4, Lemma 2.6]). *Given any projection  $p \in A$ , there is a canonical unitary isomorphism  $L^2(pAp) \cong p(L^2 A)p$  sending  $\sqrt{\phi} \in L^2(pAp)$  to  $\sqrt{\phi \circ E}$ , where  $E(a) = pap$ .  $\square$*

The bimodule  $L^2(A)$  may be characterized as follows. It is a Hilbert space  $H$  with faithful left and right actions of  $A$ , equipped with an antilinear isometric involution  $J$  and a self-dual cone  $P \subset H$  subject to the properties

- (i)  $JAJ = A'$  on  $H$ ,
- (ii)  $JcJ = c^*$  for all  $c \in Z(A)$ ,
- (iii)  $J\xi = \xi$  for all  $\xi \in P$ ,
- (iv)  $aJaJ(P) \subseteq P$  for all  $a \in A$ ,
- (v)  $\xi a = Ja^*J\xi$  for all  $\xi \in H$  and all  $a \in A$ .

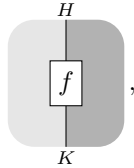
Here,  $A' := \{b \in \mathbf{B}(H) \mid [a, b] = 0, \forall a \in A\}$  is the commutant of  $A$ ;  $JAJ = \{JaJ \mid a \in A\}$ ; and the cone  $P$  is called self-dual if  $P = \{\eta \in H \mid \langle \xi, \eta \rangle \geq 0, \forall \xi \in P\}$ . The operator  $J$  is called the modular conjugation. A Hilbert space  $H$ , so equipped with a modular conjugation  $J$  and a self-dual cone  $P$ , is called a *standard form*. Such a standard form is unique up to unique unitary isomorphism [4].

*Remark 2.7.* If  $\phi$  is a faithful normal weight (an unbounded version of a state) on a von Neumann algebra  $A$ , then the GNS Hilbert space  $L^2(A, \phi)$  is a standard form for  $A$  [1], and therefore serves as a particular construction of the bimodule  $L^2(A)$ . For example, taking  $\phi$  to be the standard trace  $\text{tr}$  on  $\mathbf{B}(H)$ , we see that the ideal of Hilbert-Schmidt operators on  $H$  is a standard form for  $\mathbf{B}(H)$ .

*Example 2.8.* Let  $H$  be a Hilbert space and  $\bar{H}$  its complex conjugate. Then  $H \otimes \bar{H}$  is canonically identified with the ideal of Hilbert-Schmidt operators on  $H$ . Let  $P \subseteq H \otimes \bar{H}$  correspond to the positive Hilbert-Schmidt operators, and  $J$  to the operation  $x \mapsto x^*$ , for  $x$  a Hilbert-Schmidt operator. Then  $(H \otimes \bar{H}, J, P)$  is a standard form for  $\mathbf{B}(H)$ . We have  $J(\xi \otimes \bar{\zeta}) = \zeta \otimes \bar{\xi}$ , and  $\xi \otimes \bar{\xi} \in P$  for all  $\xi \in H$ .<sup>2</sup>

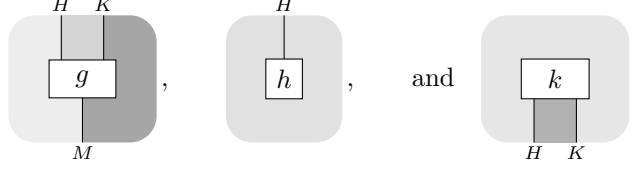
*Example 2.9.* Let  $(H, J_A, P_A)$  and  $(K, J_B, P_B)$  be standard forms for von Neumann algebras  $A$  and  $B$ . Then there is a self-dual cone  $P_{A \bar{\otimes} B}$  in  $H \otimes K$  such that  $(H \otimes K, J_A \otimes J_B, P_{A \bar{\otimes} B})$  is a standard form for  $A \bar{\otimes} B$ , and such that  $\xi \otimes \zeta \in P_{A \bar{\otimes} B}$  whenever  $\xi \in P_A$  and  $\zeta \in P_B$  [22, 27]. Note that in general  $P_{A \bar{\otimes} B}$  is strictly larger than the convex closure of  $\{\xi \otimes \zeta \mid \xi \in P_A, \zeta \in P_B\}$ .

**String diagrams.** For this paper, we have found it useful to introduce a new graphical notation, extending the well known string diagram notation used in monoidal categories and in bicategories [9, 28]. In classical string diagrams, algebras are represented by shades, bimodules are represented by lines, and homomorphisms are nodes. For example, an  $A$ - $B$ -bilinear map  $f$  between two bimodules  ${}_A H_B$  and  ${}_A K_B$  is depicted by the diagram



<sup>2</sup>Here,  $\bar{\xi} \in \bar{H}$  is the image of  $\xi \in H$  under the antilinear map  $\text{Id}_H : H \rightarrow \bar{H}$ .

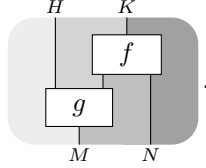
where the light shade corresponds to the algebra  $A$  and the darker shade corresponds to the algebra  $B$ . Other morphisms, such as  $g : {}_A H \boxtimes_B K_C \rightarrow {}_A M_C$ ,  $h : {}_A H_A \rightarrow {}_A L^2 A_A$ , or  $k : {}_A L^2 A_A \rightarrow {}_A H \boxtimes_B K_A$  are drawn similarly:



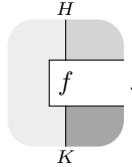
(Here,  $\boxtimes$  is the operation of Connes fusion, which will be introduced in the following section, and  ${}_A L^2 A_A$  is the identity with respect to that operation.) The identity morphism between bimodules is drawn as a single vertical line . Horizontal juxtaposition of pictures corresponds to Connes fusion, and vertical concatenation corresponds to composition of morphisms. A more complicated composition of bimodule morphisms, such as

$${}_A H \boxtimes_B K_D \xrightarrow{1_H \boxtimes f} {}_A H \boxtimes_B P \boxtimes_C N_D \xrightarrow{g \boxtimes 1_N} {}_A M \boxtimes_C N_D$$

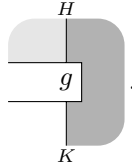
is denoted



Our addition is the introduction of a notation for morphisms that are only left-linear, or only right-linear. We denote them by nodes that extend to the right and to the left of the diagram, respectively. Thus, an  $A$ -linear morphism  $f$  between bimodules  ${}_A H_B$  and  ${}_A K_C$  is denoted



We will always use the color white for the algebra  $C$ . For example, a  $B$ -linear map  $g$  from  ${}_A H_B$  to some right  $B$ -module  $K_B$  is drawn like this:



Our conventions also allow us to speak about algebra elements using the same graphical notation, as every right (left)  $A$ -linear morphism  $L^2(A) \rightarrow L^2(A)$  is given by left (right) multiplication by an element  $a \in A$ . Such an element will be denoted , or , depending on whether we view it as acting on the left or on right on  $L^2(A)$ . The fact that an  $A$ -linear morphism  $f : {}_A H_B \rightarrow {}_A H_B$  commutes

with the left action of an element  $a \in A$  is then nicely rendered by the equation

Finally, we can also denote vectors graphically, given that an element  $\xi \in H$  is equivalent to a map  $\mathbb{C} \rightarrow H$ . For example, a vector in a bimodule  ${}_A H_B$  is denoted

The node  $\xi$  extends both to the right and to the left, as the map  $\xi : \mathbb{C} \rightarrow {}_A H_B$  is neither  $A$ - nor  $B$ -linear. Also, the space above  $\xi$  is white because the source of the above map is  ${}_{\mathbb{C}} \mathbb{C}$ .

### 3. CONNES FUSION

**Definition 3.1.** Given two modules  $H_A$  and  ${}_A K$  over a von Neumann algebra  $A$ , their Connes fusion  $H \boxtimes_A K$  is the completion of

$$(3.2) \quad \text{hom}(L^2(A)_A, H_A) \otimes_A L^2(A) \otimes_A \text{hom}({}_A L^2(A), {}_A K)$$

with respect to the inner product  $\langle \phi_1 \otimes \xi_1 \otimes \psi_1, \phi_2 \otimes \xi_2 \otimes \psi_2 \rangle := \langle (\phi_2^* \phi_1) \xi_1 (\psi_1 \psi_2^*), \xi_2 \rangle$  [3, 26, 31]. In the above equation, we have written the action of  $\psi_i$  on the right, which means that  $\psi_1 \psi_2^*$  stands for the composite  $L^2(A) \xrightarrow{\psi_1} K \xrightarrow{\psi_2^*} L^2(A)$ .

The image in the Connes fusion of an element

is equal to . Strictly speaking, the latter picture refers to the morphism

$$\mathbb{C} \xrightarrow{\xi} L^2 A \cong L^2 A \boxtimes_A L^2 A \xrightarrow{\phi \boxtimes \psi} H \boxtimes_A K,$$

but we can always identify a map from  $\mathbb{C}$  to some vector space with the vector that is the image of 1 under that map.

*Remark 3.3.* It is useful to note that the completion map from (3.2) to  $H \boxtimes_A K$  factors through both  $H \otimes_A \text{hom}({}_A L^2(A), {}_A K)$  and  $\text{hom}(L^2(A)_A, H_A) \otimes_A K$ . The Hilbert space  $H \boxtimes_A K$  therefore admits two alternative asymmetric definitions, as completions of either of those tensor products.

*Remark 3.4.* A pair of vectors  $\xi \in H_A$ ,  $\eta \in {}_A K$  does not represent anything in  $H \boxtimes_A K$ . This is nicely reflected by the fact that it is not possible to assemble the pictures and into a meaningful diagram.

**Lemma 3.5.** Let  $H$  be a Hilbert space. View its complex conjugate  $\bar{H}$  as a right  $\mathbf{B}(H)$ -module by  $\bar{\xi}a := \overline{a^* \xi}$ . Then there is a canonical isomorphism  $\bar{H} \boxtimes_{\mathbf{B}(H)} H \cong \mathbb{C}$ .

*Proof.* The Hilbert space  $L^2(\mathbf{B}(H))$  is canonically isomorphic to the space of Hilbert-Schmidt operators on  $H$ , that is to  $H \otimes \bar{H}$ ; see Example 2.8. Following Remark 3.3, the Connes fusion  $\bar{H} \boxtimes_{\mathbf{B}(H)} H$  is obtained from

$$\text{hom}((H \otimes \bar{H})_{\mathbf{B}(H)}, \bar{H}_{\mathbf{B}(H)}) \otimes_{\mathbf{B}(H)} H$$

by completing it with respect to the inner product  $\langle \phi_1 \otimes \xi_1, \phi_2 \otimes \xi_2 \rangle := \langle (\phi_2^* \phi_1) \xi_1, \xi_2 \rangle$ . There is an isomorphism

$$\begin{aligned} \bar{H} &\rightarrow \text{hom}((H \otimes \bar{H})_{\mathbf{B}(H)}, \bar{H}_{\mathbf{B}(H)}) \\ \bar{\eta} &\mapsto (\xi \otimes \bar{\zeta} \mapsto \overline{\langle \eta, \xi \rangle \zeta}) \end{aligned}$$

Applying the inverse of this isomorphism, we obtain the tensor  $\bar{H} \otimes_{\mathbf{B}(H)} H$  with the inner product  $\langle \bar{\eta}_1 \otimes \xi_1, \bar{\eta}_2 \otimes \xi_2 \rangle := \langle \eta_2 \overline{\langle \eta_1, \xi_1 \rangle}, \xi_2 \rangle = \langle \xi_1, \eta_1 \rangle \langle \eta_2, \xi_2 \rangle$ . The map  $\bar{\eta} \otimes \xi \mapsto \langle \xi, \eta \rangle$  is therefore a unitary isomorphism.  $\square$

*Remark 3.6.* The functor  $H \boxtimes_A -$  can be characterized by the existence of a right  $A$ -module isomorphism  $H \boxtimes_A L^2(A) \cong H$  (see [29]).

Connes fusion shares the formal properties of the usual algebraic tensor product:

**Proposition 3.7** ([15]). *There is a bicategory whose objects are von Neumann algebras, whose arrows are bimodules, and whose 2-morphisms are maps of bimodules. The composition of arrows is given by Connes fusion.*

Spatial tensor product of von Neumann algebras and tensor product of Hilbert spaces provides a symmetric tensor product on this bicategory, but since the formal definition of a symmetric monoidal bicategory is somewhat lengthy, we do not pursue this in detail here.

The invertible arrows of this bicategory are called *Morita equivalences*, and have the following alternative characterization:

**Proposition 3.8.** *A bimodule  ${}_A H_B$  is invertible with respect to Connes fusion if and only if the two algebras act faithfully, and  $B' := \{x \in \mathbf{B}(H) \mid [x, B] = 0\} = A$ . In that case, the inverse bimodule is given by the complex conjugate Hilbert space  $\bar{H}$ , with actions  $b\bar{\xi} := \bar{\xi}b^*$  and  $\bar{\xi}a := a^*\bar{\xi}$ .*

*Proof.* We first assume that the two actions are faithful and that  $B' = A$ . Using an  $A$ -module identification  $H \otimes \ell^2 \cong L^2 A \otimes \ell^2$ , we get isomorphisms

$$\begin{aligned} {}_A H \boxtimes_B \bar{H}_A &\cong {}_A (H \otimes \ell^2) \boxtimes_{B \otimes \mathbf{B}(\ell^2)} \overline{H \otimes \ell^2}_A \\ &\cong {}_A (L^2 A \otimes \ell^2) \boxtimes_{A \otimes \mathbf{B}(\ell^2)} \overline{L^2 A \otimes \ell^2}_A \\ &\cong {}_A L^2 A \boxtimes_A \overline{L^2 A}_A \cong {}_A L^2 A \boxtimes_A L^2 A_A \cong {}_A L^2 A_A. \end{aligned}$$

The first isomorphism follows from Lemma 3.5, and the fourth one is given by the modular conjugation on  $L^2 A$ . Similarly, we have  ${}_B \bar{H} \boxtimes_A H_B \cong {}_B L^2(B)_B$ , and so  ${}_A H_B$  is invertible. Conversely, if  ${}_A H \boxtimes_B \bar{H}_A \cong {}_A L^2 A_A$ , then the  $A$ -action is faithful and we have

$$B' \subset \text{End}({}_A H \boxtimes_B \bar{H}_A) = \text{End}({}_A L^2 A_A) = A,$$

from which  $B' = A$  follows. Similarly, the faithfulness of the right  $B$ -action follows from the isomorphism  ${}_B \bar{H} \boxtimes_A H_B \cong {}_B L^2(B)_B$ .  $\square$

**Lemma 3.9.** *Let  ${}_A H$  be a faithful  $A$ -module and let  $f : K_A \rightarrow L_A$  be an  $A$ -linear map. Then  $f$  is injective if and only if  $f \otimes 1_H : K \boxtimes_A H \rightarrow L \boxtimes_A H$  is injective.*

*Proof.* Pick an  $A$ -module identification  $H \otimes \ell^2 \cong L^2 A \otimes \ell^2$ . We then have

$$\begin{aligned}
 f \text{ is injective} &\Leftrightarrow K \boxtimes_A L^2 A \xrightarrow{f \otimes 1} L \boxtimes_A L^2 A \quad \text{is injective} \\
 &\Leftrightarrow K \boxtimes_A L^2 A \otimes \ell^2 \xrightarrow{f \otimes 1 \otimes 1} L \boxtimes_A L^2 A \otimes \ell^2 \quad \text{is injective} \\
 &\Leftrightarrow K \boxtimes_A H \otimes \ell^2 \xrightarrow{f \otimes 1 \otimes 1} L \boxtimes_A H \otimes \ell^2 \quad \text{is injective} \\
 &\Leftrightarrow K \boxtimes_A H \xrightarrow{f \otimes 1} L \boxtimes_A H \quad \text{is injective.} \quad \square
 \end{aligned}$$

#### 4. DUALIZABLE BIMODULES

A von Neumann algebra whose center is one dimensional is called a *factor*. A von Neumann algebra has finite-dimensional center if and only if it is a finite direct sum of factors. Given an  $A$ - $B$ -bimodule  $H$  over von Neumann algebras with finite center, we say that a  $B$ - $A$ -bimodule  $\bar{H}$  is dual to  $H$  if it comes equipped with maps

$$(4.1) \quad R : {}_A L^2(A)_A \rightarrow {}_A H \boxtimes_B \bar{H}_A \quad S : {}_B L^2(B)_B \rightarrow {}_B \bar{H} \boxtimes_A H_B$$

subject to the duality equations  $(R^* \otimes 1)(1 \otimes S) = 1$ ,  $(S^* \otimes 1)(1 \otimes R) = 1$ , and to the normalization condition  $R^*(pxq \otimes 1)R = S^*(1 \otimes pxq)S$  for all  $x \in \text{End}({}_A H_B)$  and all minimal central projections  $p \in Z(A)$  and  $q \in Z(B)$ . The first two conditions are classical [21]. The latter was inspired by [20, Lemma 3.9]. The above equations are best depicted using string diagrams. The duality equations are given by

$$(4.2) \quad \begin{array}{c} \text{Diagram 1: A box with a vertical line on the left and a vertical line on the right. A line from the left line goes up, loops around a box labeled $S$, and then goes down to the right line. A line from the right line goes up, loops around a box labeled $R^*$, and then goes down to the left line. This is equal to a vertical line with a shaded region on the left and an unshaded region on the right.} \end{array} \quad \text{and} \quad \begin{array}{c} \text{Diagram 2: A box with a vertical line on the left and a vertical line on the right. A line from the left line goes up, loops around a box labeled $R$, and then goes down to the right line. A line from the right line goes up, loops around a box labeled $S^*$, and then goes down to the left line. This is equal to a vertical line with a shaded region on the left and an unshaded region on the right.} \end{array}$$

and the normalization condition is

$$(4.3) \quad \begin{array}{c} \text{Diagram 3: A box with a vertical line on the left and a vertical line on the right. A line from the left line goes up, loops around a box labeled $R$, and then goes down to the right line. A line from the right line goes up, loops around a box labeled $R^*$, and then goes down to the left line. The box is labeled $p$ on the left, $x$ in the middle, and $q$ on the right. This is equal to a box with a vertical line on the left and a vertical line on the right. A line from the left line goes up, loops around a box labeled $S$, and then goes down to the right line. A line from the right line goes up, loops around a box labeled $S^*$, and then goes down to the left line. The box is labeled $p$ on the left, $x$ in the middle, and $q$ on the right.} \end{array}$$

The two shades stand for the algebras  $A$  and  $B$ , and the lines correspond to the bimodules  ${}_A H_B$  and  ${}_B \bar{H}_A$ . Note that the two sides of (4.3) are in  $p \text{End}({}_A L^2(A)_A) \cong p Z(A) \cong \mathbb{C}$  and  $q \text{End}({}_B L^2(B)_B) \cong q Z(B) \cong \mathbb{C}$ , respectively, and so it makes sense to ask for them to be equal.

**Definition 4.4.** A bimodule whose dual module exists is called *dualizable*.

We will show later, in Corollary 6.12, that the dual of a dualizable bimodule is canonically isomorphic to the complex conjugate of the bimodule. For the time being, we now reserve the notation  ${}_B \bar{H}_A$  for the dual.

*Remark 4.5.* In the literature, the term *dual* typically refers to a solution of (4.2) only. (When the conditions (4.2) are re-expressed purely in terms of  $R$  and  $S^*$  the triple  $(\bar{H}, R, S^*)$  is called a right dual, and when the conditions are re-expressed in terms of  $R^*$  and  $S$  the triple  $(\bar{H}, S, R^*)$  is called a left dual.) Such a dual, if it exists, is well defined up to unique isomorphism. However, in our Hilbert space context, having an object that is well defined up to unique isomorphism is not sufficient, as the isomorphism might fail to be unitary. Condition (4.3) is added to ensure that the dual is well defined up to unique unitary isomorphism — see Theorem 4.22.



**Lemma 4.6.** *Let  ${}_A H_B$  and  ${}_B K_A$  be irreducible bimodules. If  $H$  is dualizable, then*

$$\text{hom}_{A,A}(L^2(A), H \boxtimes_B K) \cong \begin{cases} \mathbb{C} & \text{if } {}_B \bar{H}_A \cong {}_B K_A, \\ 0 & \text{otherwise.} \end{cases}$$

*Proof.* The map  $f \mapsto (S^* \otimes 1)(1 \otimes f)$  is an isomorphism between the vector spaces  $\text{hom}({}_A L^2(A)_A, {}_A H \boxtimes_B K_A)$  and  $\text{hom}({}_B \bar{H}_A, {}_B K_A)$ .  $\square$

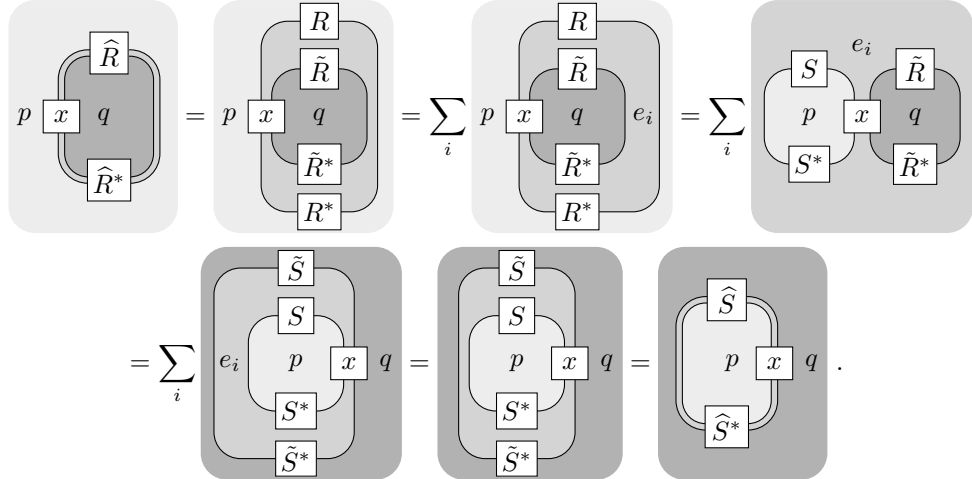
We will see later, in Lemma 4.20, that given two  $A$ - $B$ -bimodules, their direct sum is dualizable if and only if they are both dualizable. One direction is straightforward, and is given presently as Lemma 4.7. Similarly, given a non-zero  $A$ - $B$ -bimodule and a non-zero  $B$ - $C$ -bimodule, their Connes fusion is dualizable if and only if they are both dualizable. Again one direction is easier, and is given here as Lemma 4.8. The other direction is established in Corollary 7.9.

**Lemma 4.7.** *Let  ${}_A H_B$  and  ${}_A K_B$  be dualizable bimodules, with respective structure maps  $R, S, \tilde{R}$ , and  $\tilde{S}$ . Then  ${}_A(H \oplus K)_B$  is dualizable, with dual  ${}_A(\bar{H} \oplus \bar{K})_B$ , and structure maps*

$$\begin{pmatrix} R \\ 0 \\ 0 \\ \tilde{R} \end{pmatrix} : L^2(A) \rightarrow (H \oplus K) \boxtimes_B (\bar{H} \oplus \bar{K}), \quad \begin{pmatrix} S \\ 0 \\ 0 \\ \tilde{S} \end{pmatrix} : L^2(B) \rightarrow (\bar{H} \oplus \bar{K}) \boxtimes_A (H \oplus K). \quad \square$$

**Lemma 4.8.** *Let  ${}_A H_B$  and  ${}_B K_C$  be dualizable bimodules, with respective structure maps  $R, S$ , and  $\tilde{R}, \tilde{S}$ . Their fusion  ${}_A H \boxtimes_B K_C$  is then also dualizable, with dual  ${}_C \bar{K} \boxtimes_B \bar{H}_A$ , and structure maps  $\hat{R} := (1 \otimes \tilde{R} \otimes 1)R$  and  $\hat{S} := (1 \otimes S \otimes 1)\tilde{S}$ .*

*Proof.* The duality equations (4.2) for  $\hat{R}$  and  $\hat{S}$  are straightforward. To verify the normalization condition (4.3), we make use of the graphical calculus introduced earlier:



Here,  $e_i \in Z(B)$  are the minimal central projections of  $B$ . The shades correspond to the algebras  $A, B, C$ , and the lines stand for the bimodules  $H, \bar{H}, K$ , and  $\bar{K}$ .  $\square$

We henceforth often abbreviate the maps  $R : {}_A L^2(A)_A \rightarrow {}_A H \boxtimes_B \bar{H}_A$  and  $S : {}_B L^2(B)_B \rightarrow {}_B \bar{H} \boxtimes_A H_B$  as  $\text{cap}$  and  $\text{cup}$  respectively. We will show, in Theorem 4.12, that a bimodule between von Neumann algebras with finite center is “non-normalized dualizable” if and only if it is dualizable. We first record two lemmas regarding consequences of the duality equations (4.2).

**Lemma 4.9.** *Let  ${}_A H_B$  be a non-zero bimodule between factors. If  $R$  and  $S$  are maps as in (4.1) satisfying (4.2), then  $R$  and  $S$  are injective and  $(R^* R)(S^* S) \geq 1$ .*

*Proof.* The expressions  $R^*R = \text{cap}$  and  $S^*S = \text{cup}$  are in  $\mathbb{C}$  (in fact in  $\mathbb{R}$ ) because  $A$  and  $B$  are factors. As  $H$  is non-zero and  $A$  and  $B$  are factors,  $H$  is faithful, both as an  $A$ -module and a  $B^{op}$ -module. By (4.2) and Lemma 3.9, this implies that  $S$  and  $R$  are injective. In particular,  $\text{cap}$  and  $\text{cup}$  are nonzero. Letting  $e_1 := (\text{cap})^{-1} \cdot \text{cup}$  and  $e_2 := (\text{cup})^{-1} \cdot \text{cap}$  (these are the Jones projections), we have

$$e_1 = e_1 e_1 \geq e_1 e_2 e_1 = (\text{light gray circle} \cdot \text{dark gray circle})^{-1} e_1 \Rightarrow 1 \geq (\text{light gray circle} \cdot \text{dark gray circle})^{-1} \Rightarrow \text{light gray circle} \cdot \text{dark gray circle} \geq 1. \quad \square$$

The next lemma is similar to [20, Lemma 3.2].

**Lemma 4.10.** *Let  ${}_A H_B$  be a bimodule between factors. If there exist maps  $R$  and  $S$  as in (4.1) satisfying (4.2), then  ${}_A H_B$  is a finite direct sum of irreducible bimodules; its algebra of bimodule endomorphisms is therefore finite-dimensional. Moreover, the (non-normalized) state  $\varphi : \text{End}({}_A H_B) \rightarrow \mathbb{C}$  given by*

$$(4.11) \quad \varphi : x \mapsto \text{box}(x)$$

*is faithful.*

*Proof.* For any non-zero projection  $p \in \text{End}({}_A H_B)$ , we have

$$\begin{aligned}
1 = \|p\| &= \left\| \begin{array}{c} \text{[Diagram: A box labeled } p \text{ connected to a U-shaped line on a light gray background.]} \end{array} \right\| \leq \left\| \begin{array}{c} \text{[Diagram: A box labeled } p \text{ connected to a U-shaped line on a light gray background.]} \end{array} \right\| \cdot \left\| \begin{array}{c} \text{[Diagram: A U-shaped line on a dark gray background.]} \end{array} \right\| \\
&= \left\| \begin{array}{c} \text{[Diagram: A box labeled } p \text{ connected to a U-shaped line on a light gray background.]} \end{array} \right\| \cdot \left\| \begin{array}{c} \text{[Diagram: A U-shaped line on a light gray background.]} \end{array} \right\| = \sqrt{\varphi(p)} \cdot \sqrt{\bigcirc},
\end{aligned}$$

where the last step follows from the general identity  $\|a^*a\| = \|a\|^2$ . Let  $c := (\bigcirc)^{-1}$ . By the above estimate, we have  $\varphi(p) \geq c$  for any non-zero projection  $p$ . In particular,  $\varphi$  is faithful. If  $H$  failed to be a finite direct sum of irreducible bimodules, we could pick countably many non-zero mutually orthogonal projections  $p_n \in \text{End}({}_A H_B)$ , and get

$$\varphi(1) > \varphi\left(\sum_{n=1}^N p_n\right) = \sum_{n=1}^N \varphi(p_n) \geq \sum_{n=1}^N c = Nc$$

for every  $N$ . This is clearly impossible. Our bimodule is therefore a finite direct sum of irreducible ones and its endomorphism algebra is finite-dimensional.  $\square$



We can now prove that a bimodule that admits a not-necessarily normalized dual in fact admits a normalized dual:

**Theorem 4.12.** *Let  ${}_A H_B$  and  ${}_B \bar{H}_A$  be bimodules between von Neumann algebras with finite center, and let*

$$(4.13) \quad \tilde{R}: {}_A L^2(A)_A \rightarrow {}_A H \boxtimes_B \bar{H}_A \quad \text{and} \quad \tilde{S}: {}_B L^2(B)_B \rightarrow {}_B \bar{H} \boxtimes_A H_B$$

be bimodule maps satisfying (4.2). Then it is possible to find new maps  $R$  and  $S$  as in (4.1) that satisfy both (4.2) and (4.3).

*Proof.* We first assume that  $A$  and  $B$  are factors. For this proof, we write  for  $\tilde{R}$  and  for  $\tilde{S}$ , and let  $\varphi, \psi : \text{End}({}_A H_B) \rightarrow \mathbb{C}$  be given by

$\varphi : m \mapsto$  
 and  $\psi : m \mapsto$  

The new structure maps  $R$  and  $S$  are given in terms of the old ones  $\tilde{R}$  and  $\tilde{S}$  by

for some appropriately chosen positive element  $x \in \text{End}({}_A H_B)$ . Clearly  $R$  and  $S$  satisfy the duality equations (4.2). To ensure that they also satisfy the normalization equation (4.3), the element  $x$  needs to satisfy  $\varphi(xy x) = \psi(x^{-1} y x^{-1})$  for all  $y \in \text{End}({}_A H_B)$ , which is to say  $x \varphi x = x^{-1} \psi x^{-1}$  or, equivalently,  $x a x = x^{-1} b x^{-1}$ . That equation has a unique positive solution<sup>3</sup>:

When  $A = \bigoplus A_i$  and  $B = \bigoplus B_j$  are direct sums of factors, then we can write  $H$  as a direct sum of  $A_i$ - $B_j$ -bimodules  $H = \bigoplus H_{ij}$ , and similarly  $\bar{H} = \bigoplus \bar{H}_{ji}$ . We also have  $L^2A \cong \bigoplus L^2A_i$  and  $L^2B \cong \bigoplus L^2B_j$  by Lemma 2.6. The maps (4.13) induce structure maps  $\tilde{R}_{ij} : L^2A_i \rightarrow H_{ij} \boxtimes_{B_j} \bar{H}_{ji}$  and  $\tilde{S}_{ij} : L^2B_j \rightarrow \bar{H}_{ji} \boxtimes_{A_i} H_{ij}$  to which we can apply the above argument and get

$R_{ij} : {}_{A_i}L^2(A_i)_{A_i} \rightarrow {}_{A_i}H_{ij} \boxtimes_{B_j} \bar{H}_{ji} {}_{A_i}$  and  $S_{ij} : {}_{B_j}L^2(B_j)_{B_j} \rightarrow {}_{B_j}\bar{H}_{ji} \boxtimes_{A_i} H_{ij} {}_{B_j}$   
subject to (4.2) and (4.3). The desired maps  $R$  and  $S$  are then given by

and

$$L^2 B \cong \bigoplus_j L^2 B_j \xrightarrow{\bigoplus_{ij} S_{ij}} \bigoplus_{ij} (\bar{H}_{ji} \boxtimes_{A_i} H_{ij}) \subset \bigoplus_{lij} (\bar{H}_{li} \boxtimes_{A_i} H_{ij}) \cong \bar{H} \boxtimes_A H. \quad \square$$

*Remark 4.14.* We will see later, in Proposition 7.17, that when  $H$  is irreducible the mere existence of non-zero maps  $\tilde{R} : L^2(A) \rightarrow H \boxtimes_B \tilde{H}$  and  $\tilde{S} : L^2(B) \rightarrow \tilde{H} \boxtimes_A H$  implies that  ${}_A H_B$  is dualizable.

We now discuss two lemmas that we will need in order to prove, in Theorem 4.22, that the dual is well defined up to unique unitary isomorphism.

**Lemma 4.15.** *Let  $A$  and  $B$  be factors, and let  ${}_A H_B$  be a dualizable bimodule, with structure maps  $R$  and  $S$  subject to (4.2) and (4.3). Then the state  $\varphi$  defined in (4.11) is a trace.*

<sup>3</sup>Courtesy of <http://mathoverflow.net/questions/70838>.

*Proof.* By a few applications of (4.2) and some planar isotopies, we get

$$(4.16) \quad \begin{array}{c} \text{Diagram 1} \\ \text{Diagram 2} \end{array} = \begin{array}{c} \text{Diagram 3} \\ \text{Diagram 4} \end{array} \quad \text{and} \quad \begin{array}{c} \text{Diagram 5} \\ \text{Diagram 6} \end{array} = \begin{array}{c} \text{Diagram 7} \\ \text{Diagram 8} \end{array}.$$

Combining these equations with (4.3) yields

$$\begin{array}{c} \text{Diagram 9} \\ \text{Diagram 10} \end{array} = \begin{array}{c} \text{Diagram 11} \\ \text{Diagram 12} \end{array} = \begin{array}{c} \text{Diagram 13} \\ \text{Diagram 14} \end{array} = \begin{array}{c} \text{Diagram 15} \\ \text{Diagram 16} \end{array} = \begin{array}{c} \text{Diagram 17} \\ \text{Diagram 18} \end{array}.$$

The latter being true for any  $y \in \text{End}({}_A H_B)$  and the state  $\varphi$  being faithful by Lemma 4.10, it follows that

$$\hat{x} := \begin{array}{c} \text{Diagram 19} \\ \text{Diagram 20} \end{array} = \begin{array}{c} \text{Diagram 21} \\ \text{Diagram 22} \end{array}.$$

Equivalently, the map  $x \mapsto \hat{x}$  is an involution.

As in the proof of the previous lemma, pick a trace  $\tau$  and a positive invertible element  $a$  such that  $\varphi = a\tau$ . Our goal is to show that  $\varphi$  is a trace; this is true provided  $a$  is central. Equation (4.16) implies  $a\hat{x} = xa$  for all  $x$ . Equivalently, we have  $\hat{x} = a^{-1}xa$ . Because the map  $x \mapsto \hat{x}$  is an involution, we have  $x = \hat{\hat{x}} = a^{-2}xa^2$ . Since  $a$  is positive and its square is central,  $a$  is also central.  $\square$

As a corollary of the above proof, we see

$$(4.17) \quad \begin{array}{c} \text{Diagram 23} \\ \text{Diagram 24} \end{array} = \begin{array}{c} \text{Diagram 25} \\ \text{Diagram 26} \end{array}, \quad \text{and thus also} \quad \bar{x} := \begin{array}{c} \text{Diagram 27} \\ \text{Diagram 28} \end{array} = \begin{array}{c} \text{Diagram 29} \\ \text{Diagram 30} \end{array}.$$

*Remark 4.18.* The first equation in (4.17) is essentially the same as [8, Theorem 4.1.18], which states that the  $n$ th power of the operation

$$(4.19) \quad \rho_n : \underbrace{\begin{array}{c} \text{Diagram 31} \\ \text{Diagram 32} \end{array}}_n \mapsto \begin{array}{c} \text{Diagram 33} \\ \text{Diagram 34} \end{array}$$

is the identity. One should note that Jones' rotation  $\rho_n$  does not always agree with our way of interpreting figure (4.19). It agrees when the type  $II_1$  subfactor  $A \subset B$  is extremal, that is, when the normalized traces on  $A'$  and  $B$  coincide on  $A' \cap B$  or, equivalently, when the minimal conditional expectation  $B \rightarrow A$  is equal to the trace preserving one. See also Warning 5.11.

**Lemma 4.20.** *Let  ${}_A H_B$  be a dualizable bimodule with dual  ${}_B \bar{H}_A$ , and let  $p \in \text{End}({}_A H_B)$  be a projection. The  $A$ - $B$ -bimodule  $pH$  is then dualizable and its dual*



least one  $tr_{pq}$  such that  $tr_{pq}(y) \neq 0$ . Letting  $\bar{x}$  be as in (4.17), equation (4.23) implies

$$tr_{pq}(v^*v\bar{x}) = tr_{pq}(v^{-1}v^{*-1}\bar{x}) \quad \forall x \in \text{End}({}_A H_B).$$

This being true for all  $p, q$ , it follows that  $v^*v = v^{-1}v^{*-1}$ . In other words,  $v^*v = (v^*v)^{-1}$ . Since  $v^*v$  is positive, we must have  $v^*v = 1$ .  $\square$

## 5. STATISTICAL DIMENSION AND MINIMAL INDEX

The following definition is well known. Our approach follows [20].

**Definition 5.1.** If  $A$  and  $B$  are factors, then the *statistical dimension* of a dualizable bimodule  ${}_A H_B$  is given by :

$$\dim({}_A H_B) := R^*R = S^*S \in \mathbb{R}_{\geq 0}.$$

For non-dualizable bimodules, one simply declares  $\dim({}_A H_B)$  to be  $\infty$ .

The basic properties of the statistical dimension can be found in many places [13, 14, 16, 20]. We include some proofs for completeness.

**Proposition 5.2.** *The statistical dimension of a non-zero bimodule  ${}_A H_B$  is always  $\geq 1$ , and is equal to 1 if and only if  $H$  is invertible. The statistical dimension is additive under direct sums, and multiplicative under Connes fusion<sup>4</sup>. It is also multiplicative under external tensor product. In other words, we have:*

$$(5.3) \quad \dim({}_A H_B) \in \{0\} \cup [1, \infty], \text{ and it is 0 iff } H = 0$$

$$(5.4) \quad \dim({}_A H_B) = 1 \text{ iff } A' = B$$

$$(5.5) \quad \dim({}_A (H \oplus K)_B) = \dim({}_A H_B) + \dim({}_A K_B)$$

$$(5.6) \quad \dim({}_A H \boxtimes_B K_C) = \dim({}_A H_B) \dim({}_B K_C)$$

$$(5.7) \quad \dim({}_A H_B \otimes_{\mathbb{C}} ({}_C K_D)) = \dim({}_A H_B) \dim({}_C K_D)$$

*Proof.* *i.* If  $H \neq 0$ , then  $\dim({}_A H_B) \geq 1$  by Lemma 4.9. If  $H = 0$ , then clearly  $R^*R = 0$ .

*ii.* Let  $e_1, e_2$  be as in Lemma 4.9. If  $\dim({}_A H_B) = 1$ , then  $e_1 = e_1 e_2 e_1$  and  $e_2 = e_2 e_1 e_2$ . As  $e_1$  and  $e_2$  are projections, the first equation implies  $e_2 \geq e_1$ , while the second implies  $e_1 \geq e_2$ . Thus  $e_1 = e_2$ . From this (and a reflection along a vertical axis of the argument so far), we get  $\begin{array}{c} \text{---} \end{array} \begin{array}{c} \text{---} \end{array} = \begin{array}{c} \text{---} \end{array} \begin{array}{c} \text{---} \end{array} = \begin{array}{c} \text{---} \end{array} \begin{array}{c} \text{---} \end{array}$ . As  $A$  is a factor and  ${}_A H_B \neq 0$ , the latter is a faithful  $A$ -module. Lemma 4.9 implies that the projection  $RR^* = \begin{array}{c} \text{---} \end{array}$  is non-trivial. Thus, the previous equation implies  $\begin{array}{c} \text{---} \end{array} = \begin{array}{c} \text{---} \end{array}$ . The map  $R$  is therefore invertible, and similarly for  $S$ . Having shown  ${}_B \tilde{H} \boxtimes_A H_B \cong L^2 B$  and  ${}_A H \boxtimes_B \tilde{H}_A \cong L^2 A$ , the result follows from Proposition 3.8.

Conversely, if  $H$  is invertible, there exist unitary maps  $\tilde{R}: L^2(A) \rightarrow H \boxtimes_B \tilde{H}$  and  $S: L^2(B) \rightarrow \tilde{H} \boxtimes_A H$ . Since  ${}_A H_B$  is irreducible,  $\lambda := (\tilde{R}^* \otimes 1)(1 \otimes S)$  is a scalar, and so  $R := \lambda \tilde{R}$  and  $S$  satisfy (4.2). Again because  ${}_A H_B$  is irreducible (and  $R$  and  $S$  are unitary), the normalization condition (4.3) is satisfied as well. Thus  $d = R^*R = 1$ .

*iii.* If either  $H$  or  $K$  is not dualizable, then both sides of (5.5) are infinite by Lemma 4.20. If they are both dualizable, then Lemma 4.7 provides a description of the duality maps for  $H \oplus K$ , which we can use to compute

$$\dim(H \oplus K) = \begin{pmatrix} R^* \\ 0 \\ 0 \\ \tilde{R} \end{pmatrix}^* \begin{pmatrix} R \\ 0 \\ 0 \\ \tilde{R} \end{pmatrix} = R^*R + \tilde{R}^*\tilde{R} = \dim(H) + \dim(K).$$

<sup>4</sup>For this to always be true, it is appropriate to use the convention  $0 \cdot \infty = 0$ .

iv. If both  $H$  and  $K$  are dualizable, then using the duality maps described in Lemma 4.8, we compute:

$$\dim(H \boxtimes_B K) = R^*(1 \otimes \tilde{R}^* \otimes 1)(1 \otimes \tilde{R} \otimes 1)R = R^* \dim(K)R = \dim(H) \dim(K).$$

If either  $H$  or  $K$  is zero, then the equation clearly holds. The remaining case  $H \neq 0$ ,  $\dim(K) = \infty$  requires different techniques<sup>5</sup> and will be treated later, in Corollary 7.9.

v. Apply equation (5.6) to the decomposition

$$({}_A H_B) \otimes_{\mathbb{C}} ({}_C K_D) \cong (({}_A H_B) \otimes_{\mathbb{C}} ({}_C L^2 C_C)) \boxtimes_{B \otimes C} (({}_B L^2 B_B) \otimes_{\mathbb{C}} ({}_C K_D)). \quad \square$$

*Remark 5.8.* As was shown in the celebrated papers [7, 12], equation (5.3) can be improved: the statistical dimension of a bimodule can only take values in the set  $\{2 \cos(\frac{\pi}{n}); n = 2, 3, 4, \dots\} \cup [2, \infty]$ .

If the von Neumann algebras  $A = \bigoplus A_i$  and  $B = \bigoplus B_j$  are finite direct sums of factors (in other words have finite centers), then any  $A$ - $B$ -bimodule  $H$  can be written as a direct sum

$$(5.9) \quad H = \bigoplus H_{ij}$$

of  $A_i$ - $B_j$ -bimodules. The statistical dimension of  ${}_A H_B$  is then best taken to be a matrix of numbers:

$$\dim({}_A H_B)_{ij} := \dim({}_{A_i} H_{ij} {}_{B_j}).$$

The matrix-valued statistical dimension satisfies the same formal properties (5.3)–(5.7) as above, provided the right hand sides of (5.6) and (5.7) are interpreted in terms of matrix and Kronecker products, respectively.

As will be shown later, in Corollary 7.14, the following definition of minimal index is equivalent to other definitions that exist in the literature [12, 13, 16, 24]:

**Definition 5.10.** The *minimal index*  $[B : A]$  of an inclusion of factors  $\iota : A \rightarrow B$  is the square of the statistical dimension of  ${}_A L^2 B_B$ .

*Warning 5.11.* The above definition does not always agree with Jones' original definition [7]. It agrees if and only if the type  $II_1$  subfactor  $A \subset B$  is extremal, that is, the normalized traces on  $A'$  and  $B$  coincide on  $A' \cap B$ .

Let  $\iota : A \rightarrow B$  be a subfactor. If the minimal index  $[B : A]$  is finite, we say that  $\iota$  is a finite homomorphism. More generally, if  $A$  and  $B$  are von Neumann algebras with finite center, we say that a homomorphism  $A \rightarrow B$  is finite if all the matrix entries of  $\dim({}_A L^2 B_B)$  are finite. Of course, this simply amounts to the following definition:

**Definition 5.12.** A homomorphism  $A \rightarrow B$  between von Neumann algebras with finite center is *finite* if the associated bimodule  ${}_A L^2 B_B$  is dualizable.

When dealing with inclusions of von Neumann algebras with finite center, the matrix  $\dim({}_A L^2 B_B)$  is much better behaved than the corresponding matrix of minimal indices. We propose a new notation for it:

**Definition 5.13.** Given a finite homomorphism  $f : A \rightarrow B$  between von Neumann algebras with finite center, we let  $\llbracket B : A \rrbracket := \dim({}_A L^2 B_B)$  denote the matrix of statistical dimensions of  ${}_A L^2 B_B$ .

---

<sup>5</sup>Note that the special case  $\dim(H) = 1$ ,  $\dim(K) = \infty$  is straightforward, as fusing with an invertible bimodule certainly doesn't change the property of having a dual or not.

Following (5.6), the matrix of statistical dimensions satisfies

$$(5.14) \quad \llbracket B : A \rrbracket \llbracket C : B \rrbracket = \llbracket C : A \rrbracket.$$

As an corollary of Lemma 4.10, we have:

**Lemma 5.15.** *Let  $f : A \rightarrow B$  be a finite homomorphism between von Neumann algebras with finite center. Then the relative commutant of  $f(A)$  in  $B$  is finite-dimensional.*

*Proof.* The relative commutant of  $f(A)$  in  $B$  is the endomorphism algebra of the bimodule  ${}_A L^2(B)_B$ . Apply Lemma 4.10 to every summand in the decomposition (5.9) of that bimodule.  $\square$

**Lemma 5.16.** *Let  ${}_A H_B$  be a bimodule between von Neumann algebras with finite center. Assume  $B$  acts faithfully, and let  $B' \supset A$  be the commutant of  $B$  on  $H$ . Then  $\dim({}_A H_B) = \llbracket B' : A \rrbracket$ .*

*Proof.* The bimodule  ${}_{B'} H_B$  is a Morita equivalence, and its matrix of statistical dimensions is therefore an identity matrix. We have

$$\dim({}_A H_B) = \dim({}_A L^2 B' \boxtimes_{B'} H_B) = \dim({}_A L^2 B'_{B'}) \dim({}_{B'} H_B) = \dim({}_A L^2 B'_{B'}).$$

The last expression is the definition of the matrix  $\llbracket B' : A \rrbracket$ .  $\square$

**Corollary 5.17.** *If  $A \subset B \subset \mathbf{B}(H)$  are von Neumann algebras with finite center, then  $\llbracket B : A \rrbracket = \llbracket A' : B' \rrbracket^T$ . In particular, if  $A$  and  $B$  are factors, then  $\llbracket B : A \rrbracket = \llbracket A' : B' \rrbracket$ .*

*Proof.* Let  $\overline{H}$  denote the complex conjugate of  $H$ , with actions as in Proposition 3.8. Applying Lemma 5.16 twice, we have  $\llbracket B : A \rrbracket = \dim({}_A H_{B'}) = \dim({}_{B'} \overline{H}_A)^T = \llbracket A' : B' \rrbracket^T$ .  $\square$

**Lemma 5.18.** *Let  $B$  be a factor, and let  $A \subset B$  be a subalgebra with finite center. Call its minimal central projections  $p_1, \dots, p_n$ . Then  $\sum [p_i B p_i : p_i A] = \|\llbracket B : A \rrbracket\|^2$ , where  $\|\cdot\|$  stands for the  $\ell^2$ -norm of a vector.*

*Proof.* The  $i$ th entry in the vector  $\llbracket B : A \rrbracket = \dim({}_A L^2 B_B)$  is by definition

$$\dim({}_{p_i A} (p_i L^2 B)_B) = \dim({}_{p_i A} (p_i L^2 B p_i)_{p_i B p_i}) = \dim({}_{p_i A} L^2 (p_i B p_i)_{p_i B p_i}),$$

where the first equality holds because  ${}_B (L^2 B p_i)_{p_i B p_i}$  is an invertible bimodule, and the second one follows from Lemma 2.6. Therefore,  $\dim({}_{p_i A} (p_i L^2 B)_B)^2 = [p_i B p_i : p_i A]$ . The results now follows by summing over all the indices  $i$ .  $\square$

For more results about statistical dimension and minimal index, we refer the reader to [10, 12, 13, 14, 18, 19].

## 6. FUNCTORIALITY OF THE $L^2$ -SPACE AND OF CONNES FUSION

**The inner product on  $L^2(A)$ .** We mentioned earlier that for a von Neumann algebra  $A$ , its  $L^2$ -space is a completion of the vector space  $\bigoplus_{\phi \in L^1_+(A)} \mathbb{C} \sqrt{\phi}$  with respect to some pre-inner product. To define  $\langle \sqrt{\phi}, \sqrt{\psi} \rangle$ , one considers the function  $f(t) := \phi([D\phi : D\psi]_t)$ , where  $[D\phi : D\psi]_t \in A$  denotes Connes' Radon-Nikodym derivative<sup>6</sup>. The function  $f$  can be analytically continued from  $\mathbb{R}$  to the strip  $\Im m(t) \in [0, 1]$ , and the value of the inner product is then given by  $f(i/2)$ :

$$(6.1) \quad \langle \sqrt{\phi}, \sqrt{\psi} \rangle := \text{anal. cont. } \phi([D\phi : D\psi]_t)_{t \rightarrow i/2}.$$

<sup>6</sup>We work with a definition of the Radon-Nikodym derivative  $[D\phi : D\psi]_t$  that does not require  $\phi$  and  $\psi$  to be faithful; it satisfies  $[D\phi : D\psi]_t \in \mathfrak{s}_\phi A \mathfrak{s}_\psi$  where  $\mathfrak{s}_\phi$  and  $\mathfrak{s}_\psi$  are the support projections of  $\phi$  and  $\psi$ .



In particular, we have  $\|\sqrt{\phi}\|^2 = \phi(1)$ .

The cone of positive elements in  $L^2 A$  is given by  $L_+^2(A) := \{\sqrt{\phi} \mid \phi \in L_+^1(A)\}$ , and the two actions of  $A$  on  $L^2 A$  are prescribed by

$$\langle a\sqrt{\phi}b, \sqrt{\psi} \rangle := \text{anal. cont. } \phi([D\phi : D\psi]_t \sigma_t^\psi(b)a),$$

where  $\sigma_t^\psi$  is the modular flow<sup>7</sup>. The space  $L^2 A$  is also equipped with the modular conjugation  $J_A$ , that sends  $\lambda\sqrt{\phi}$  to  $\bar{\lambda}\sqrt{\phi}$  for  $\lambda \in \mathbb{C}$ , and satisfies

$$(6.2) \quad J_A(a\xi b) = b^* J_A(\xi) a^*.$$

Altogether, the triple  $(L^2(A), J_A, L_+^2(A))$  is a standard form for the von Neumann algebra  $A$ ; compare [3, p.528].

The above constructions are compatible with spatial tensor product in the sense that there is a natural isomorphism  $L^2(A \bar{\otimes} B) \cong L^2(A) \otimes L^2(B)$  that respects the left and right  $A \bar{\otimes} B$ -actions, and intertwines the modular involutions — see Example 2.9.

*Remark 6.3* (The modular algebra). The construction of  $L^2 A$  is best understood in the larger context of the modular algebra [23, 32] — recall Remark 2.4. The modular algebra is

$$L^* A := \bigoplus_{p \in \mathbb{C}_{\Re \geq 0}^\times \cup \{\infty\}} L^p A,$$

and can be represented as an algebra of unbounded operators on a Hilbert space. The product sends  $L^p(A) \times L^q(A)$  to  $L^{\frac{1}{1/p+1/q}}(A)$ , and  $L^\infty(A)$  is a synonym for  $A$ . Given  $p \in \mathbb{C}_{\Re \geq 0}^\times$ , then for every  $\phi \in L_+^1 A$ , its  $p$ th root  $\phi^{1/p}$  (in the sense of functional calculus) belongs to  $L^p A$ . In particular, we have  $\sqrt{\phi} \equiv \phi^{1/2} \in L^2(A)$ . The modular conjugation  $J_A : L^2(A) \rightarrow L^2(A)$  is then simply the restriction of the  $*$ -operation on  $L^* A$ . There is also a faithful normal trace  $Tr : L^* A \rightarrow \mathbb{C}$  given by

$$Tr(\phi) = \begin{cases} \phi(1) & \text{for } \phi \in L^1(A) \\ 0 & \text{for } \phi \in L^p(A), \quad p \neq 1. \end{cases}$$

By definition, it satisfies  $Tr(\phi a) = \phi(a)$  for  $\phi \in L^1 A$  and  $a \in A$ .

Using complex exponentiation in the algebra  $L^* A$ , the Radon-Nikodym derivative and the modular flow can be recovered<sup>8</sup> as

$$(6.4) \quad \begin{aligned} [D\phi : D\psi]_t &= \phi^{it} \psi^{-it} \\ \sigma_t^\psi(a) &= \psi^{it} a \psi^{-it} \end{aligned} \quad (t \in \mathbb{R}).$$

We can therefore rewrite the quantity that appears in the right hand side of (6.1) as

$$\phi([D\phi : D\psi]_t) = Tr(\phi[D\phi : D\psi]_t) = Tr(\phi \phi^{it} \psi^{-it}) = Tr(\phi^{1+it} \psi^{-it}).$$

The last expression  $Tr(\phi^{1+it} \psi^{-it})$  can be evaluated for any  $t$  in the strip  $\Im m(t) \in [0, 1]$ , because  $\Re(1+it)$  and  $\Re(-it)$  are both non-negative there. Moreover, the dependence on  $t$  is analytic by [32, Corollary 2.6]. One can therefore rewrite the inner product on  $L^2(A)$  as

$$\langle \sqrt{\phi}, \sqrt{\psi} \rangle = Tr(\phi^{1+it} \psi^{-it})|_{t=i/2} = Tr(\phi^{1/2} \psi^{1/2}),$$

<sup>7</sup>We do not assume that  $\psi$  is faithful in defining the modular flow  $\sigma_t^\psi$ . For  $a \in A$ , we have  $\sigma_t^\psi(a) \in \mathbf{s}_\psi A \mathbf{s}_\psi$ .

<sup>8</sup>Unfortunately, one cannot use (6.4) to define  $[D\phi : D\psi]_t$  and  $\sigma_t^\psi$ , as the Radon-Nikodym derivative and the modular flow are needed for the construction of the modular algebra — see [32].

and the fact that it is symmetric follows from the trace property. The inner product also admits the following alternative definition:

$$\langle \sqrt{\phi}, \sqrt{\psi} \rangle := \text{anal. cont. } \lim_{t \rightarrow -i/2} \psi([D\phi : D\psi]_t).$$

This definition agrees with definition (6.1) because  $\psi([D\phi : D\psi]_t) = \text{Tr}(\psi \phi^{it} \psi^{-it}) = \text{Tr}(\phi^{it} \psi^{1-it})$  and  $\text{Tr}(\phi^{it} \psi^{1-it})|_{t=-i/2} = \text{Tr}(\phi^{1/2} \psi^{1/2})$ .

We will need the following lemma later on in order to identify the dual of the bimodule  ${}_A L^2 B_B$  associated to a finite homomorphism  $A \rightarrow B$ .

**Lemma 6.5.** *Let  $\{p_i \in A\}$  be orthogonal projections adding up to 1. If  $\phi \in L_+^1(A)$  satisfies  $p_i \phi p_j = 0$  for all  $i$  and  $j$  with  $i \neq j$ , then  $p_i \sqrt{\phi} p_j = 0$  for all  $i$  and  $j$  with  $i \neq j$ .*

*Proof.* Applying functional calculus to an (unbounded) operator in block diagonal form yields an operator in block diagonal form. The result follows since the modular algebra has a representation by unbounded operators [32], and  $\sqrt{\phi}$  is obtained from  $\phi$  by functional calculus.  $\square$

In our analysis of conditional expectations in section 7, we will use the following general fact relating Radon-Nikodym derivatives in different algebras — see [2, Lemma 1.4.4] and [5, Theorem 4.7]. Let  $A \subset B$  be a subalgebra, and let  $E : B \rightarrow A$  be a faithful completely positive normal map such that  $E(axb) = aE(x)b$  for  $x \in B$ ,  $a, b \in A$ ; in this case,

$$(6.6) \quad [D(\phi \circ E) : D(\psi \circ E)]_t = [D\phi : D\psi]_t.$$

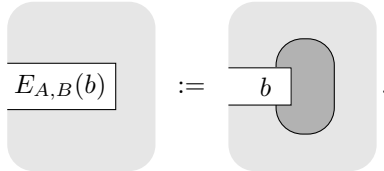
**Functoriality of the  $L^2$ -space.** The following theorem is closely related to some known results [10, 14]. Nevertheless, it appears to be new:

**Theorem 6.7.** *The assignment  $A \mapsto L^2(A)$  defines a functor from the category of von Neumann algebras with finite-dimensional center and finite homomorphisms, to the category of Hilbert spaces and bounded linear maps.*

*Proof.* Given a finite homomorphism  $A \rightarrow B$  between von Neumann algebras with finite center, let  $E_{A,B} : B \rightarrow A$  be the map given by

$$(6.8) \quad E_{A,B}(b)\xi := R^*(b \otimes 1)R\xi, \quad \xi \in L^2(A),$$

where  $R : {}_A L^2(A)_A \rightarrow {}_A L^2(B)_B \boxtimes_B \overline{L^2(B)}_A$  is as in (4.1), and the  $b$  that appears in the right hand side of (6.8) acts by left multiplication on  $L^2(B)$ . Graphically, this is:



As before, the two shades represent the algebras  $A$  and  $B$ , and the lines stand for the bimodule  ${}_A L^2 B_B$  and its dual. The fact that the box labeled  $E_{A,B}(b)$  extends to the left of the picture refers to the fact that the map  $E_{A,B}(b) : {}_A L^2 A_A \rightarrow {}_A L^2 A_A$  is only right  $A$ -linear.

The map (6.8) satisfies  $E_{A,B}(aba') = aE_{A,B}(b)a'$  for any  $a, a' \in A$  and  $b \in B$ . Moreover, for every sequence  $A \rightarrow B \rightarrow C$  of composable arrows, we have

$$(6.9) \quad E_{A,B} \circ E_{B,C} = E_{A,C}$$

by Lemma 4.8. The map  $L^2(f) : L^2(A) \rightarrow L^2(B)$  associated to the finite homomorphism  $f : A \rightarrow B$  is then defined by

$$(6.10) \quad L^2(f) : \sqrt{\phi} \mapsto \sqrt{\phi \circ E_{A,B}}.$$

To see that this map is well defined and bounded, we exhibit a constant  $C$  such that

$$\left\| \sum_j c_j \sqrt{\phi_j \circ E_{A,B}} \right\|^2 \leq C \cdot \left\| \sum_j c_j \sqrt{\phi_j} \right\|^2 \quad \forall c_j \in \mathbb{C}, \phi_j \in L_+^1(A).$$

Let  $\{p_\alpha\}$  be the minimal central projections of  $A$ . Since  $E_{A,B}(1)$  is central, we can write it as  $E_{A,B}(1) = \sum_\alpha C_\alpha p_\alpha$  for some given constants  $C_\alpha$ . Let

$$C := \max_\alpha C_\alpha = \|E_{A,B}(1)\|.$$

Using the shorthand notation  $\phi_{j,\alpha} := \phi_j p_\alpha$ , we then have

$$\begin{aligned} \left\| \sum_j c_j \sqrt{\phi_j \circ E_{A,B}} \right\|^2 &= \sum_{j,k} c_j \bar{c}_k \langle \sqrt{\phi_j \circ E_{A,B}}, \sqrt{\phi_k \circ E_{A,B}} \rangle \\ &= \sum_{j,k} c_j \bar{c}_k \text{anal. cont.}_{t \rightarrow i/2} \phi_j \circ E_{A,B} ([D(\phi_j \circ E_{A,B}) : D(\phi_k \circ E_{A,B})]_t) \\ &= \sum_{j,k} c_j \bar{c}_k \text{anal. cont.}_{t \rightarrow i/2} \phi_j \circ E_{A,B} ([D\phi_j : D\phi_k]_t) \\ &= \sum_{j,k} c_j \bar{c}_k \text{anal. cont.}_{t \rightarrow i/2} \phi_j (E_{A,B}(1) [D\phi_j : D\phi_k]_t) \\ &= \sum_{\alpha,j,k} c_j \bar{c}_k \text{anal. cont.}_{t \rightarrow i/2} \phi_{j,\alpha} (C_\alpha [D\phi_{j,\alpha} : D\phi_{k,\alpha}]_t) \\ &= \sum_\alpha C_\alpha \left\| \sum_j c_j \sqrt{\phi_{j,\alpha}} \right\|^2 \leq C \cdot \sum_\alpha \left\| \sum_j c_j \sqrt{\phi_{j,\alpha}} \right\|^2 = C \cdot \left\| \sum_j c_j \sqrt{\phi_j} \right\|^2, \end{aligned}$$

where the third equality follows from (6.6) and the fourth one follows from the  $A$ -linearity of  $E_{A,B}$ .

The compatibility of (6.10) with composition follows from (6.9).  $\square$

*Remark 6.11.* Given a finite homomorphism  $f : A \rightarrow B$  between von Neumann algebras with finite center, one can also define  $L^p(f) : L^p A \rightarrow L^p B$ ;  $\phi^{1/p} \mapsto (\phi \circ E_{A,B})^{1/p}$ . These assemble to a  $*$ -algebra homomorphism  $\bigoplus L^p A \rightarrow \bigoplus L^p B$ ; see [32, section 3].

**Corollary 6.12.** *Let  ${}_A H_B$  be a bimodule between von Neumann algebras with finite center. Then its dual bimodule, if it exists, is canonically isomorphic to the complex conjugate Hilbert space, with actions given by  $b\xi a := a^* \xi b^*$ .*

*Proof.* Let  ${}_A H_B$  be dualizable. By Lemma 4.10 and the decomposition (5.9), this bimodule is a finite direct sum of irreducible bimodules. Both duals and complex conjugates being compatible with the direct sum operation, it is enough to treat the irreducible case. We assume for simplicity that the action  $\rho : A \rightarrow \mathbf{B}(H)$  is faithful. The general case follows.

Let  ${}_B \overline{H}_A^c$  denote the complex conjugate of  ${}_A H_B$ , and let  $B'$  be the commutant of  $B$  on  $H$ . By Proposition 3.8, we have  ${}_{B'} H \boxtimes_B \overline{H}_{B'}^c \cong {}_{B'} L^2(B')_{B'}$ , and so

$$\begin{aligned} {}_A H \boxtimes_B \overline{H}_A^c &\cong {}_A L^2(B') \boxtimes_{B'} H \boxtimes_B \overline{H}^c \boxtimes_{B'} L^2(B')_A \\ &\cong {}_A L^2(B') \boxtimes_{B'} L^2(B') \boxtimes_{B'} L^2(B')_A \cong {}_A L^2(B')_A. \end{aligned}$$

By Theorem 6.7, we therefore get a map  ${}_A L^2(A)_A \rightarrow {}_A H \boxtimes_B \overline{H}_A^c$  which is non-trivial by construction — see for instance equation (6.20). The result now follows from Lemma 4.6.  $\square$

*Remark 6.13.* The isomorphism between any dual and the complex conjugate bimodule constructed in the proof of Corollary 6.12 is in fact unitary. We do not include a proof — see Proposition 6.16 for a related result.

In the special case of the bimodule  ${}_A L^2(B)_B$  associated to a finite homomorphism  $f : A \rightarrow B$ , together with a chosen dual  $({}_B \overline{L^2 B}_A, R, S)$ , the isomorphism  $\overline{L^2 B} \cong \overline{L^2 B}^c$  is given by

$$\begin{aligned} \overline{L^2 B} &\cong \overline{L^2 B} \boxtimes_A L^2 A \xrightarrow{1 \otimes L^2(f)} \overline{L^2 B} \boxtimes_A L^2 B \cong \overline{L^2 B} \boxtimes_A L^2 B \boxtimes_B L^2 B \\ &\xrightarrow{1 \otimes 1 \otimes J} \overline{L^2 B} \boxtimes_A L^2 B \boxtimes_B \overline{L^2 B}^c \xrightarrow{S^* \otimes 1} L^2 B \boxtimes_B \overline{L^2 B}^c \cong \overline{L^2 B}^c, \end{aligned}$$

where  $J$  is the modular conjugation. This isomorphism  $\overline{L^2 B} \cong \overline{L^2 B}^c$  is chosen so as to make the composite

$$\begin{aligned} {}_A L^2(A)_A &\xrightarrow{R} {}_A L^2(B) \boxtimes_B \overline{L^2(B)}_A \\ &\cong {}_A L^2(B) \boxtimes_B \overline{L^2(B)}^c_A \xrightarrow{1 \otimes J} {}_A L^2(B) \boxtimes_B L^2(B)_A \cong {}_A L^2(B)_A \end{aligned}$$

equal to  $L^2(f)$ .

Instead of identifying the dual of  ${}_A L^2 B_B$  with  ${}_B \overline{L^2 B}_A$ , we can identify it with  ${}_B L^2 B_A$ , as follows. There is an isomorphism  $\Phi$  between any dual of  ${}_A L^2(B)_B$  and  ${}_B L^2(B)_A$  given by

$$\begin{aligned} (6.14) \quad {}_B \overline{L^2 B}_A &\cong {}_B \overline{L^2 B} \boxtimes_A L^2 A \xrightarrow{1 \otimes L^2(f)} {}_B \overline{L^2 B} \boxtimes_A L^2 B_A \\ &\cong {}_B \overline{L^2 B} \boxtimes_A L^2 B \boxtimes_B L^2 B_A \xrightarrow{S^* \otimes 1} {}_B L^2 B \boxtimes_B L^2 B \cong {}_B L^2 B_A. \end{aligned}$$

In graphical notation we have

$$\Phi := \begin{array}{c} \begin{array}{c} \overline{{}_B L^2 B}_A \\ \downarrow \\ \text{[Diagram: A box labeled } L^2(f) \text{ with a box labeled } S^* \text{ below it, connected by a line. The box } S^* \text{ is inside a larger box labeled } {}_B L^2 B_A \text{ at the bottom.}] \\ \downarrow \\ {}_B L^2 B_A \end{array} \end{array}.$$

The isomorphism  $\Phi$  makes the following diagram commutative:

$$(6.15) \quad \begin{array}{ccc} {}_A L^2 A_A & \xrightarrow{R} & {}_A L^2(B) \boxtimes_B \overline{L^2(B)}_A \\ L^2(f) \downarrow & & \downarrow 1 \otimes \Phi \\ {}_A L^2 B_A & \xrightarrow{\cong} & {}_A L^2(B) \boxtimes_B L^2(B)_A \end{array}.$$

**Proposition 6.16.** *Let  $f : A \rightarrow B$  be a finite homomorphism, and let  $(\overline{L^2 B}, R, S)$  be a chosen dual to the bimodule  ${}_A L^2 B_B$  associated to  $f$ . The isomorphism  $\Phi := (S^* \otimes 1)(1 \otimes L^2(f))$  from  ${}_B \overline{L^2 B}_A$  to  ${}_B L^2 B_A$  is unitary.*

*Proof.* The algebra  $B \cap A' = \text{End}({}_A L^2 B_B)$  is finite-dimensional by Lemma 4.10 and decomposition (5.9). Let  $p_1, \dots, p_n \in B \cap A'$  be mutually orthogonal minimal projections adding up to 1, and let  $\bar{p}_i := (S^* \otimes 1)(1 \otimes p_i \otimes 1)(1 \otimes R)$  be the dual projection defined in equation (4.21). Let  $E : B \rightarrow A$  be as in (6.8). For every  $i \neq j$  and  $\phi \in L_+^1 A$ , the element  $p_i(\phi \circ E)p_j \in L^1(B)$  is zero, as  $(p_i(\phi \circ E)p_j)(b) = \phi \circ E(p_j b p_i)$  and

$$E(p_j b p_i) = \begin{array}{c} \text{[Diagram: A box labeled } p_i \text{ at the top, a box labeled } b \text{ in the middle, and a box labeled } p_j \text{ at the bottom, all inside a larger box.]} \\ = \begin{array}{c} \text{[Diagram: A box labeled } b \text{ at the top, a box labeled } p_j \text{ in the middle, and a box labeled } \bar{p}_i \text{ at the bottom, all inside a larger box.]} \\ = \begin{array}{c} \text{[Diagram: A box labeled } b \text{ at the top, a box labeled } p_j \text{ in the middle, and a box labeled } p_i \text{ at the bottom, all inside a larger box.]} \\ = 0. \end{array} \end{array}$$

It follows from Lemma 6.5 that  $p_i \sqrt{\phi \circ E} p_j = 0$  for  $i \neq j$ . The map  $L^2(f) : \sqrt{\phi} \mapsto \sqrt{\phi \circ E}$  therefore factors as

$$\begin{array}{ccc} L^2(f) : L^2 A & \longrightarrow & L^2(B) \cong \bigoplus_{ij} p_i L^2(B) p_j \\ & \searrow \text{dashed} & \uparrow \\ & \bigoplus_{[L^2(f)]_i} & \bigoplus_i p_i L^2(B) p_i \end{array}.$$

We have a similar factorization of  $R$  by Lemma 4.7:

$$\begin{array}{ccc} R : L^2 A & \longrightarrow & L^2(B) \boxtimes_B \overline{L^2(B)} \cong \bigoplus_{ij} p_i L^2(B) \boxtimes_B \bar{p}_j \overline{L^2(B)} \\ & \searrow \text{dashed} & \uparrow \\ & \bigoplus_{R_i} & \bigoplus_i p_i L^2(B) \boxtimes_B \bar{p}_i \overline{L^2(B)} \end{array}.$$

Let us write

$$\Phi_{jk} : \bar{p}_j \overline{L^2(B)} \rightarrow L^2(B) p_k$$

for the components of  $\Phi$ . Given that  ${}_B \bar{p}_j \overline{L^2(B)}_A$  and  ${}_B L^2(B) p_k{}_A$  are irreducible bimodules, the maps  $\Phi_{jk}$  are either zero or a scalar multiple of some unitary. By the commutativity of (6.15) (and since  $R_i \neq 0$ ), the subspace  $\bigoplus_i p_i L^2(B) \boxtimes_B \bar{p}_i \overline{L^2(B)}$  of  $L^2(B) \boxtimes_B \overline{L^2(B)}$  goes to  $\bigoplus_i p_i L^2(B) p_i$  under the map

$$1 \otimes \Phi : \bigoplus_{ij} p_i L^2(B) \boxtimes_B \bar{p}_j \overline{L^2(B)} \rightarrow \bigoplus_{ik} p_i L^2(B) \boxtimes_B L^2(B) p_k \cong \bigoplus_{ik} p_i L^2 B p_k.$$

It follows that  $\Phi_{jk} = 0$  whenever  $j \neq k$ . We can therefore rewrite  $\Phi$  as

$$\Phi = \bigoplus_i \Phi_i : \bigoplus_i \bar{p}_i \overline{L^2(B)} \rightarrow \bigoplus_i L^2(B) p_i,$$

where each  $\Phi_i$  is a scalar multiple of some unitary.

To finish the argument, we show that each  $\Phi_i$  has norm 1. Let  $q_i \in Z(A') = Z(A)$  be the central support projection of  $p_i \in A'$ . The maps

$$\begin{aligned} [L^2(f)]_i : {}_A L^2 A_A &\rightarrow {}_A p_i L^2(B) p_i{}_A \quad \text{and} \\ R_i : {}_A L^2 A_A &\rightarrow {}_A p_i L^2(B) \boxtimes_B \bar{p}_i \overline{L^2(B)}_A \end{aligned}$$

factor through  ${}_A q_i (L^2 A)_A$ , and are therefore scalar multiples of partial isometries. Given  $\phi \in q_i (L^2_+ A)$ , we have

$$\begin{aligned} \|[L^2(f)]_i(\sqrt{\phi})\|^2 &= \|p_i L^2(f)(\sqrt{\phi})\|^2 = \|p_i \sqrt{\phi \circ E}\|^2 \\ &= \langle p_i \sqrt{\phi \circ E}, \sqrt{\phi \circ E} \rangle = \phi \circ E(p_i) = E(p_i) \cdot \phi(1), \end{aligned}$$

where  $E(p_i) \in q_i Z(A) \cong \mathbb{C}$ . Similarly, we have

$$\begin{aligned} \|R_i(\sqrt{\phi})\|^2 &= \|p_i R(\sqrt{\phi})\|^2 \\ &= \left\langle \overline{\begin{array}{|c|} \hline \sqrt{\phi} \\ \hline \boxed{p_i} \end{array}}, \overline{\begin{array}{|c|} \hline \sqrt{\phi} \\ \hline \boxed{p_i} \end{array}} \right\rangle = \overline{\begin{array}{|c|} \hline \sqrt{\phi} \\ \hline \boxed{p_i} \end{array}} \cdot \begin{array}{|c|} \hline \sqrt{\phi} \\ \hline \boxed{p_i} \end{array} = \overline{\begin{array}{|c|} \hline \sqrt{\phi} \\ \hline \boxed{p_i} \end{array}} \cdot \langle \sqrt{\phi}, \sqrt{\phi} \rangle = E(p_i) \cdot \phi(1). \end{aligned}$$

It follows that  $\|R_i\|^2 = \|[L^2(f)]_i\|^2 = E(p_i)$ . Since (6.15) is commutative, we thus get  $\|\Phi_i\| = \|[L^2(f)]_i\|/\|R_i\| = 1$ , and the map  $\Phi = \bigoplus_i \Phi_i$  is therefore unitary.  $\square$

The reader may wonder whether the condition of finite center was really needed in Theorem 6.7. We saw in Theorem 4.12 that a bimodule between von Neumann algebras with finite center is dualizable if and only if there exist maps  $R$  and  $S$  satisfying (4.2): though a priori dualizability requires both conditions (4.2) and (4.3), in fact it is detected by condition (4.2) only. If the centers of  $A$  and  $B$  are not atomic (that is, if one of them contains  $L^\infty([0, 1])$ ), then we do not know how to formulate (4.3). We therefore do not have a good notion of duality in that context;

however, we may still define a homomorphism  $f : A \rightarrow B$  between arbitrary von Neumann algebras to be finite if there exist maps  $R$  and  $S$  satisfying (4.2), that is giving a not-necessarily normalized dual for the bimodule  ${}_A L^2 B_B$ .

**Conjecture 6.17.** *The assignment  $A \mapsto L^2 A$  extends to a functor from the category of all von Neumann algebras and finite homomorphisms to the category of Hilbert spaces and bounded linear maps.*

The following two lemmas describe how the functor  $L^2$  interacts with the basic operations of taking corner and block-diagonal subalgebras. Recall from Lemma 2.6 that the  $L^2$ -space of the corner algebra  $A_0 := pAp$  is given by  $L^2(A_0) = p(L^2 A)p$ .

**Lemma 6.18.** *Let  $f : A \rightarrow B$  be a finite homomorphism between von Neumann algebras with finite center. Given a projection  $p \in A$ , let  $A_0 := pAp$ ,  $B_0 := pBp$ , and  $f_0 := f|_{A_0} : A_0 \rightarrow B_0$ , where we identify  $p$  with its image  $f(p) \in B$ . Then the homomorphism  $f_0$  is finite, and we have*

$$L^2(f_0) = L^2(f)|_{L^2(A_0)},$$

where we have identified  $L^2(A_0)$  and  $L^2(B_0)$  with the subspaces  $pL^2(A)p$  and  $pL^2(B)p$  of  $L^2(A)$  and  $L^2(B)$  respectively.

*Proof.* The structure maps (4.1) for the dual of  ${}_A L^2(B)_B$  restrict to maps

$$\begin{aligned} R_0 : {}_{A_0} L^2 A_0 {}_{A_0} &= {}_{A_0} p L^2 A p {}_{A_0} \rightarrow {}_{A_0} p L^2 B \boxtimes_B \overline{L^2 B} p {}_{A_0} = {}_{A_0} p L^2 B p \boxtimes_{B_0} \overline{p L^2 B p} {}_{A_0}, \\ S_0 : {}_{B_0} L^2 B_0 {}_{B_0} &= {}_{B_0} p L^2 B p {}_{B_0} \rightarrow {}_{B_0} p \overline{L^2 B} \boxtimes_A L^2 B p {}_{B_0} = {}_{B_0} p \overline{L^2 B} p \boxtimes_{A_0} p L^2 B p {}_{B_0}. \end{aligned}$$

Here we use the invertibility of  ${}_B L^2 B p {}_{B_0}$  to rewrite the targets of  $R_0$  and  $S_0$ . These satisfy the duality equations (4.2) and the normalization (4.3), and therefore exhibit  ${}_{B_0} (p \overline{L^2(B)} p) {}_{A_0}$  as the dual of  ${}_{A_0} L^2(B_0) {}_{B_0}$ . For every  $b \in B_0$ , we have

$$\begin{aligned} E_{A,B}(b)\xi &= R^*(b \otimes 1)R\xi = R_0^*(b \otimes 1)R_0\xi = E_{A_0,B_0}(b)\xi \quad \text{for } \xi \in L^2(A_0), \\ E_{A,B}(b)\xi &= R^*(b \otimes 1)R\xi = R^*(pb \otimes 1)R(1-p)\xi = 0 \quad \text{for } \xi \in L^2(A_0)^\perp, \end{aligned}$$

from which it follows that  $E_{A_0,B_0} = E_{A,B}|_{B_0}$ . Given a state  $\phi : A_0 \rightarrow \mathbb{C}$ , the image of  $L^2(f_0)(\sqrt{\phi})$  in  $L^2(B)$  is the square root of

$$b \mapsto \phi(E_{A_0,B_0}(pbp)) = \phi(E_{A,B}(pbp)) = \phi(pE_{A,B}(b)p),$$

and is thus equal to the image of  $\sqrt{a \mapsto \phi(pap)}$  under  $L^2(f)$ .  $\square$

**Lemma 6.19.** *Let  $A$  be a factor, and  $p_1, \dots, p_n \in A$  be a collection of orthogonal projections that add up to 1. Let  $\iota : \bigoplus p_i A p_i \rightarrow A$  denote the inclusion. Then  $L^2(\iota)$  is the natural inclusion*

$$L^2(\bigoplus p_i A p_i) \cong \bigoplus p_i L^2(A) p_i \hookrightarrow L^2(A),$$

where the first isomorphism is given by Lemma 2.6. In particular,  $L^2(\iota)$  is an isometry.

*Proof.* We write  $A_i$  for  $p_i A p_i$ . The inclusions

$$\begin{aligned} R : {}_{\bigoplus A_i} L^2(\bigoplus A_i) {}_{\bigoplus A_i} &\cong \bigoplus p_i L^2(A) p_i \hookrightarrow L^2(A) \cong {}_{\bigoplus A_i} L^2(A) \boxtimes_A L^2(A) {}_{\bigoplus A_i} \\ S : {}_A L^2(A) {}_A &\hookrightarrow \bigoplus_i L^2(A) \cong \bigoplus_i L^2(A) p_i \boxtimes_{\bigoplus A_i} p_i L^2(A) \cong {}_A L^2(A) \boxtimes_{\bigoplus A_i} L^2(A) {}_A \end{aligned}$$

exhibit  ${}_{\bigoplus A_i} L^2(A) {}_A$  as the dual of  ${}_A L^2(A) {}_{\bigoplus A_i}$ . For  $\xi_i \in L^2(A_i)$  and  $a \in A$ , equation (6.8) reads

$$\bigoplus_i \xi_i \xrightarrow{R} \sum_i \xi_i \xrightarrow{a} \sum_i a \xi_i \xrightarrow{R^*} \bigoplus_j p_j (\sum_i a \xi_i) p_j = \bigoplus_i p_i a p_i \xi_i = (\bigoplus_i p_i a p_i) (\bigoplus_i \xi_i).$$

The map  $E := E_{\oplus A_i, A}$  is therefore given by  $E(a) = \bigoplus q_i(a)$ , where  $q_i(a) := p_i a p_i$ . It follows that  $L^2(\iota)(\sqrt{\bigoplus \phi_i}) = \sqrt{(\bigoplus \phi_i) \circ E} = \sqrt{\sum \phi_i \circ q_i} = \sum \sqrt{\phi_i \circ q_i}$ . Since  $\sqrt{\phi_i} \in L^2(A_i)$  maps to  $\sqrt{\phi_i \circ q_i} \in L^2(A)$  under the map described in Lemma 2.6, this finishes the proof.  $\square$

One drawback of the construction presented in Theorem 6.7 is that the maps  $L^2(f) : L^2(A) \rightarrow L^2(B)$  are not isometric. For example, if  $\iota : A \rightarrow B$  is a finite map between factors, then  $L^2(\iota)$  is  $\sqrt[4]{[B : A]} = \sqrt{[B : A]}$  times an isometry. This can be checked on positive vectors: since  $\|\sqrt{\phi}\|^2 = \phi(1)$  and  $\|\sqrt{\phi \circ E_{A,B}}\|^2 = \phi(E_{A,B}(1)) = E_{A,B}(1)\phi(1)$ , it follows that

$$(6.20) \quad \|L^2(\iota)(\sqrt{\phi})\|/\|\sqrt{\phi}\| = \sqrt{E_{A,B}(1)} = \sqrt{R^*R} = \sqrt{\dim({}_A L^2 B_B)}$$

for any  $\sqrt{\phi} \in L^2_+(A)$ . In some sense, that is inevitable. Assuming that  $\iota$  is injective, let  $L^2(\iota)_{\text{iso}}$  denote the isometry in the polar decomposition of  $L^2(\iota)$ . The assignment

$$(6.21) \quad (\iota : A \rightarrow B) \mapsto (L^2(\iota)_{\text{iso}} : L^2(A) \rightarrow L^2(B))$$

is not a functor — this issue is already visible with finite-dimensional commutative von Neumann algebras. Nevertheless, we have:

**Proposition 6.22.** *When restricted to the subcategory of von Neumann algebras with finite center and injective finite homomorphisms  $\iota : A \rightarrow B$  that satisfy  $Z(B) \subset \iota(A)$ , the assignment  $\iota \mapsto L^2(\iota)_{\text{iso}}$  is a functor.*

*Proof.* We can write  $\iota : A \rightarrow B$  as a direct sum of maps  $\iota_j : A_j \rightarrow B_j$ , where each  $B_j$  is a factor. Let us decompose each  $A_j$  as a direct sum of factors  $A_j = \bigoplus_i A_{ij}$ , where  $A_{ij} = p_{ij} A_j$ , and  $p_{ij}$  are the minimal central projections of  $A_j$ . We can then factor  $\iota$  as

$$\iota : A = \bigoplus_{ij} A_{ij} \rightarrow \bigoplus_{ij} p_{ij} B_j p_{ij} \rightarrow \bigoplus_j B_j = B.$$

Applying the functor  $L^2$  (as defined in Theorem 6.7) to the above maps, we get

$$L^2(\iota) : L^2(A) = \bigoplus_{ij} L^2(A_{ij}) \rightarrow \bigoplus_{ij} L^2(p_{ij} B_j p_{ij}) \xrightarrow{\star} \bigoplus_j L^2(B_j) = L^2(B).$$

The map  $\star$  is an isometry by Lemma 6.19. The isometry  $L^2(\iota)_{\text{iso}}$  is therefore the composite of  $L^2_{\text{iso}} : \bigoplus L^2(A_{ij}) \rightarrow \bigoplus L^2(p_{ij} B_j p_{ij})$  with the natural inclusion  $\bigoplus L^2(p_{ij} B_j p_{ij}) \hookrightarrow \bigoplus L^2(B_j)$  described in Lemma 2.6.

Given two composable inclusions  $\iota : A \rightarrow B$  and  $\kappa : B \rightarrow C$  with  $Z(B) \subset \iota(A)$  and  $Z(C) \subset \kappa(B)$ , we now show that  $L^2(\kappa \circ \iota)_{\text{iso}} = L^2(\kappa)_{\text{iso}} \circ L^2(\iota)_{\text{iso}}$ . Let us write  $C = \bigoplus C_k$ ,  $B = \bigoplus B_{jk}$ , and  $A = \bigoplus A_{ijk}$  as sums of factors, where  $\iota(A_{ijk}) \subset B_{jk}$  and  $\kappa(B_{jk}) \subset C_k$ . The corresponding minimal central projections are denoted  $p_{ijk} \in A_{ijk}$  and  $q_{jk} \in B_{jk}$ . To compare  $L^2(\kappa \circ \iota)_{\text{iso}}$  with  $L^2(\kappa)_{\text{iso}} \circ L^2(\iota)_{\text{iso}}$ , we consider the following diagram

$$\begin{array}{ccccc} \bigoplus_{jk} L^2(B_{jk}) & \xrightarrow{L^2_{\text{iso}}} & \bigoplus_{jk} L^2(q_{jk} C q_{jk}) & \hookrightarrow & \bigoplus_k L^2(C_k) \\ \uparrow & & \uparrow & \nearrow & \\ \bigoplus_{ijk} L^2(p_{ijk} B p_{ijk}) & \xrightarrow{L^2_{\text{iso}}} & \bigoplus_{ijk} L^2(p_{ijk} C p_{ijk}) & & \\ L^2_{\text{iso}} \uparrow & \nearrow L^2_{\text{iso}} & & & \\ \bigoplus_{ijk} L^2(A_{ijk}) & & & & \end{array}$$

The upper right triangle is a diagram of inclusions and commutes for obvious reasons. The upper left rectangle commutes by the functoriality of the  $L^2$  construction (Theorem 6.7) and by the compatibility of polar decomposition with the operation of composing with an isometry. Finally, note that whenever we have a subfactor

inclusion  $f : N \rightarrow M$  then, by equation (6.20), the corresponding map  $L^2(f)$  is a scalar multiple of an isometry. The commutativity of the bottom triangle thus holds because  $A_{ijk} \hookrightarrow p_{ijk} B p_{ijk} \hookrightarrow p_{ijk} C p_{ijk}$  are subfactor inclusions.  $\square$

**Functoriality of Connes fusion.** By construction, the operation of Connes fusion  $(H_A, {}_A K) \mapsto H \boxtimes_A K$  is a functor in  $H$  and  $K$ . We now investigate in what sense it is a functor of the *three* variables  $H$ ,  $A$ , and  $K$ . Consider the following category. Its objects are triples  $(H, A, K)$  consisting of a von Neumann algebra  $A$  with finite center, a right module  $H$ , and a left module  $K$ . A morphism from  $(H_1, A_1, K_1)$  to  $(H_2, A_2, K_2)$  is a triple  $\alpha : A_1 \rightarrow A_2$ ,  $h : H_1 \rightarrow H_2$ ,  $k : K_1 \rightarrow K_2$ , where  $\alpha$  is a finite homomorphism, and  $h$  and  $k$  are  $A_1$ -linear maps.

**Theorem 6.23.** *The assignment*

$$(H, A, K) \mapsto H \boxtimes_A K$$

*extends to a functor from the category described above to the category of Hilbert spaces and bounded linear maps.*

*Proof.* Given a morphism  $(h, \alpha, k) : (H_1, A_1, K_1) \rightarrow (H_2, A_2, K_2)$  of the above category, we describe the induced map  $h \boxtimes_{\alpha} k : H_1 \boxtimes_{A_1} K_1 \rightarrow H_2 \boxtimes_{A_2} K_2$ . Recall that the composite (6.14) provides an isomorphism  $\Phi$  between the dual of the bimodule  ${}_A L^2(A_2)_{A_2}$  and the bimodule  ${}_A L^2(A_2)_{A_1}$ . Let

$$\begin{aligned} R &: {}_{A_1} L^2(A_1)_{A_1} \rightarrow {}_{A_1} L^2(A_2) \boxtimes_{A_2} \overline{L^2(A_2)}_{A_1} \xrightarrow{1 \otimes \Phi} {}_{A_1} L^2(A_2) \boxtimes_{A_2} L^2(A_2)_{A_1} \\ S &: {}_{A_2} L^2(A_2)_{A_2} \rightarrow \overline{L^2(A_2)}_{A_2} \boxtimes_{A_1} L^2(A_2)_{A_2} \xrightarrow{\Phi \otimes 1} {}_{A_2} L^2(A_2) \boxtimes_{A_1} L^2(A_2)_{A_2} \end{aligned}$$

denote the composition of the normalized duality maps (4.1) with the aforementioned isomorphism.

We define the image of an element

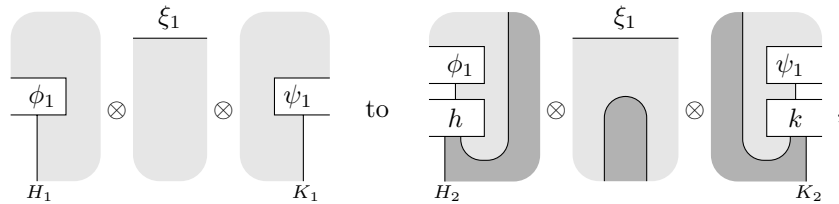
$$\phi_1 \otimes \xi_1 \otimes \psi_1 \in \text{hom}_{A_1}(L^2 A_1, H_1) \otimes L^2 A_1 \otimes \text{hom}_{A_1}(L^2 A_1, K_1)$$

under the map  $h \boxtimes_{\alpha} k$  to be  $\phi_2 \otimes \xi_2 \otimes \psi_2$ , where  $\phi_2 \in \text{hom}_{A_2}(L^2 A_2, H_2)$  and  $\psi_2 \in \text{hom}_{A_2}(L^2 A_2, K_2)$  are given by

$$\begin{aligned} \phi_2 : L^2 A_2 \cong L^2 A_1 \boxtimes_{A_1} L^2 A_2 &\xrightarrow{\phi_1 \otimes 1} H_1 \boxtimes_{A_1} L^2 A_2 \xrightarrow{h \otimes 1} H_2 \boxtimes_{A_1} L^2 A_2 \\ &\cong H_2 \boxtimes_{A_2} L^2 A_2 \boxtimes_{A_1} L^2 A_2 \xrightarrow{1 \otimes S^*} H_2 \boxtimes_{A_2} L^2 A_2 \cong H_2 \end{aligned}$$

$$\begin{aligned} \psi_2 : L^2 A_2 \cong L^2 A_2 \boxtimes_{A_1} L^2 A_1 &\xrightarrow{1 \otimes \psi_1} L^2 A_2 \boxtimes_{A_1} K_1 \xrightarrow{1 \otimes k} L^2 A_2 \boxtimes_{A_1} K_2 \\ &\cong L^2 A_2 \boxtimes_{A_1} L^2 A_2 \boxtimes_{A_2} K_2 \xrightarrow{S^* \otimes 1} L^2 A_2 \boxtimes_{A_2} K_2 \cong K_2, \end{aligned}$$

and  $\xi_2 := R(\xi_1) \in L^2(A_2) \boxtimes_{A_2} L^2(A_2) \cong L^2(A_2)$ . Note that  $\xi_2 = L^2(\alpha)(\xi_1)$  by diagram (6.15). Graphically, the above map sends





and is therefore given by

$$(6.24) \quad h \boxtimes_{\alpha} k : \begin{array}{c} \xi_1 \\ \hline \begin{array}{|c|c|} \hline \phi_1 & \psi_1 \\ \hline \end{array} \\ \hline \begin{array}{|c|c|} \hline H_1 & K_1 \\ \hline \end{array} \end{array} \mapsto \begin{array}{c} \xi_1 \\ \hline \begin{array}{|c|c|} \hline \phi_1 & \psi_1 \\ \hline \end{array} \\ \hline \begin{array}{|c|c|} \hline h & k \\ \hline \end{array} \\ \hline \begin{array}{|c|c|} \hline H_2 & K_2 \\ \hline \end{array} \end{array} = \begin{array}{c} \xi_1 \\ \hline \begin{array}{|c|c|} \hline \phi_1 & \psi_1 \\ \hline \end{array} \\ \hline \begin{array}{|c|c|} \hline h & k \\ \hline \end{array} \\ \hline \begin{array}{|c|c|} \hline H_2 & K_2 \\ \hline \end{array} \end{array}.$$

Here, the two shades correspond to the algebras  $A_1$  and  $A_2$ , the unlabeled line between those shades corresponds to the bimodule  ${}_{A_1}L^2(A_2)_{A_2}$  and its dual bimodule  ${}_{A_2}L^2(A_2)_{A_1}$ , and the isomorphism (6.14) has been suppressed from the notation. Abstracting out  $\xi_1$ ,  $\phi_1$ ,  $\psi_1$  from (6.24), we can rewrite  $h \boxtimes_{\alpha} k$  in a more concise form, as

$$h \boxtimes_{\alpha} k = \begin{array}{c} H_1 \quad K_1 \\ \hline \begin{array}{|c|c|} \hline h & k \\ \hline \end{array} \\ \hline H_2 \quad K_2 \end{array}.$$

The latter description also makes it clear that  $h \boxtimes_{\alpha} k$  is bounded. Compatibility with composition follows from Lemma 4.8.  $\square$

We record the following lemma for future use. Once again, we make implicit use of the identification (6.14) and of its basic property (6.15).

**Lemma 6.25.** *Let  $f : A \rightarrow B$  be a finite map between von Neumann algebras with finite center. Then the map  $B \rightarrow \text{hom}(L^2 A_A, L^2 B_A)$  given by*

$$b \mapsto (b \otimes 1)L^2(f) = \begin{array}{c} \text{[Diagram: A box labeled } b \text{ with a U-shaped line passing through it, representing the map } (b \otimes 1)L^2(f) \text{]} \end{array}$$

is an isomorphism.

*Proof.* The inverse map is  $\begin{array}{c} \text{[Diagram: A box labeled } x \text{ with a U-shaped line passing through it, representing an element in } \text{hom}(L^2 B_B, L^2 B_B) \text{]} \end{array} \mapsto \begin{array}{c} \text{[Diagram: A box labeled } x \text{ with a U-shaped line passing through it, representing an element in } \text{hom}(L^2 B_B, L^2 B_B) \text{]} \end{array} \in \text{hom}(L^2 B_B, L^2 B_B) \cong B.$   $\square$

## 7. MINIMAL INDEX VIA CONDITIONAL EXPECTATIONS

In this section, we recall the work of Pimsner and Popa on conditional expectations, and use it to establish the equivalence between our definition of minimal index (Definitions 5.1 and 5.10) and other notions of minimal index that exist in the literature [12, 13, 16, 24]. The basic inequality (7.4) was introduced in [24] for type II von Neumann algebras, and later in [14, 16, 17] for arbitrary von Neumann algebras. Further references include [25, section 1.1] and [13, section 3.4].

Given a subfactor  $N \subset M$ <sup>9</sup>, a completely positive normal map  $E : M \rightarrow N$  is called a *conditional expectation* if  $E(1) = 1$  and  $E(axb) = aE(x)b$  for all  $x \in M$  and  $a, b \in N$ . It may happen that, for some  $\lambda$ , the conditional expectation satisfies the *Pimsner–Popa inequality*:

$$E(x) \geq \lambda^{-1}x, \quad \forall x \in M_+.$$

<sup>9</sup>Note that in this section, we usually (but not always) use the letters  $N$  and  $M$  to refer to factors, as is traditional, and use the letters  $A$  and  $B$  to refer to more general von Neumann algebras.

Following [16], the index of the conditional expectation is the smallest possible such  $\lambda$ :

$$(7.1) \quad \text{Ind}(E) := \inf \{ \lambda \mid E(x) \geq \lambda^{-1}x, \forall x \in M_+ \}.$$

We call a conditional expectation finite if its index is finite. For subfactors admitting finite conditional expectations, Longo proves [16, Theorem 5.5] that there exists a unique conditional expectation minimizing  $\text{Ind}(E)$  — see also [14]. For a general subfactor he sets

$$(7.2) \quad \text{Ind}(N, M) := \inf_E \text{Ind}(E) = \inf_E \inf \{ \lambda \mid E(x) \geq \lambda^{-1}x, \forall x \in M_+ \},$$

where the infimum runs over all conditional expectations  $E : M \rightarrow N$ . Let us call (7.2) the *Longo index* of the subfactor  $N \subset M$ . We will show later, in Corollary 7.14, that the minimal index (Definition 5.10) agrees with the Longo index for infinite-dimensional factors — see Warning 7.16.

If the subfactor has finite minimal index, then an example of a conditional expectation is given by  $[M : N]^{-\frac{1}{2}}$  times the map (6.8):

$$E_0(b) := [M : N]^{-\frac{1}{2}} \cdot \text{[diagram of } b \text{ with a cap]} = (\text{[diagram of a cap]})^{-1} \cdot \text{[diagram of } b \text{ with a cap]}.$$

We call  $E_0$  the *minimal conditional expectation*. We will show later, in Proposition 7.10, that the minimal conditional expectation minimizes  $\text{Ind}(E)$ , thus justifying its name.

We begin by observing that the minimal index of a subfactor provides an upper bound on the index of the minimal conditional expectation:

**Proposition 7.3.** *The minimal conditional expectation  $E_0$  satisfies the inequality*

$$(7.4) \quad E_0(x) \geq [M : N]^{-1}x \quad \forall x \in M_+.$$

*In other words,  $\text{Ind}(E_0) \leq [M : N]$ .*

*Proof.* Let  $x$  be a positive element of  $M$ , and let us write  $d := [M : N]^{\frac{1}{2}}$  for the statistical dimension of  ${}_N L^2 M_M$ . Because the map  $d^{-1} \text{[diagram of a cap]}$  is a projection, we have  $d^{-1} \text{[diagram of a cap]} \leq \text{[diagram of a cap]}$ . As a consequence of the general fact  $(a \leq b) \Rightarrow (yay^* \leq yby^*)$ , it follows that

$$d^{-1} \text{[diagram of } x \text{ with a cap]} = d^{-1} \text{[diagram of } \sqrt{x} \text{ with a cap]} = \text{[diagram of } \sqrt{x} \text{ with a cap]} \leq \text{[diagram of } \sqrt{x} \text{ with a cap]} = \text{[diagram of } x \text{ with a cap]}.$$

Now multiply both sides by  $d^{-1}$  to get the desired inequality.  $\square$

The following proposition establishes the connection between the Pimsner–Popa inequality and dualizability.

**Proposition 7.5.** *Let  $A \subset B$  be von Neumann algebras with finite center, and let  $E : B \rightarrow A$  be a conditional expectation. If there exists a constant  $\mu > 0$  such that  $E(x) \geq \mu x$  for all  $x \in B_+$ , then  ${}_A L^2 B_B$  is a dualizable bimodule.*

*Proof.* We show that  ${}_B L^2 B_A$  is the dual of  ${}_A L^2 B_B$ . To do so, we construct maps

$$(7.6) \quad \begin{aligned} R &= \text{[Diagram: a cap with a dot]} : {}_A L^2(A)_A \rightarrow {}_A L^2 B_A \cong {}_A L^2(B) \boxtimes_B L^2(B)_A \\ S &= \text{[Diagram: a cup with a dot]} : {}_B L^2(B)_B \rightarrow {}_B L^2(B) \boxtimes_A L^2(B)_B \end{aligned}$$

satisfying the duality equations (4.2), and appeal to Theorem 4.12 in order to achieve the normalization (4.3).

Using equation (6.6) we see that the map  $R$  defined by  $\sqrt{\phi} \mapsto \sqrt{\phi \circ E}$  is an isometry. Let  $e := RR^* = \text{[Diagram: a cap with a dot]} be the corresponding Jones projection. By [25, Theorem 1.1.6], there exists a set of elements  $b_j \in B$  such that  $\{b_j e b_j^*\}$  are mutually orthogonal projections forming a partition of unity, and such that  $\sum b_j b_j^* \in B$  is a bounded operator. Here, both  $b_j$  and  $b_j^*$  refer to left multiplication operators on  $L^2 B$ . It follows that the map  $\sum b_j : \bigoplus_j L^2(B) \rightarrow L^2(B)$  is also bounded. Let  $K$  be the right  $A$ -module  $\bigoplus_j L^2 A$ , and let  $m$  and  $\bar{m}$  be the following two maps:$

$$m : K \boxtimes_A L^2 B \cong \bigoplus_j L^2(B) \xrightarrow{\sum (b_j \cdot)} L^2 B, \quad \bar{m} : L^2 B \boxtimes_A \bar{K} \cong \bigoplus_j L^2(B) \xrightarrow{\sum (\cdot b_j)} L^2 B.$$

Graphically, the equation  $\sum b_j e b_j^* = 1$  means that the map

$$\text{[Diagram: a vertical tube with a cap labeled } m^* \text{ at the top and a cup labeled } m \text{ at the bottom, connected by a dotted line]} : L^2(B) \rightarrow L^2(B)$$

is the identity, where the dotted line stands for  $K$ . It is then easy to check that, along with  $R$ , the map

$$S := \text{[Diagram: a vertical tube with a cap labeled } m^* \text{ at the top and a cup labeled } \bar{m} \text{ at the bottom, connected by a dotted line]} = \text{[Diagram: a vertical tube with a cap labeled } m^* \text{ at the top and a cup labeled } m \text{ at the bottom]} = \text{[Diagram: a vertical tube with a cap labeled } \bar{m}^* \text{ at the top and a cup labeled } \bar{m} \text{ at the bottom]}$$

satisfies the duality equations (4.2).  $\square$

The above proof also shows that the following variant of Proposition 7.5 holds.

**Proposition 7.7.** *Let  $f : A \rightarrow B$  be a map between arbitrary von Neumann algebras, and let  $E : B \rightarrow A$  be a conditional expectation such that  $E(x) \geq \mu x$  for all  $x \in B_+$ . Then  $f$  is a finite homomorphism in the sense (see the discussion before Conjecture 6.17) that  ${}_A L^2 B_B$  admits a not-necessarily normalized dual bimodule.  $\square$*

As a first application of the Pimsner–Popa inequality, we have:

**Lemma 7.8.** *Let  $N \subset P \subset M$  be factors. Then  $[M : N] < \infty \Rightarrow [P : N] < \infty$ .*

*Proof.* Let  $E : M \rightarrow N$  be the minimal conditional expectation. Then  $E|_P$  is a conditional expectation subject to the same bound:  $E|_P(x) \geq [M : N]^{-1}x, \forall x \in P_+$ . The subfactor  $N \subset P$  satisfies the condition of Proposition 7.5, and so  ${}_N L^2 P_P$  is dualizable.  $\square$

**Corollary 7.9.** *Let  $N, P, M$  be factors, and let  ${}_N H_P$  and  ${}_P K_M$  be non-zero bimodules. If their fusion  ${}_N H \boxtimes_P K_M$  is a dualizable  $N$ - $M$ -bimodule, then  ${}_N H_P$  and  ${}_P K_M$  are dualizable.*

*Proof.* We show that  ${}_N H_P$  is dualizable. Let  $P'$  be the commutant of  $P$  on  $H$ , and let  $M'$  be the commutant of  $M$  on  $H \boxtimes_P K$ . We have  $N \subset P' \subset M'$ . By Lemma 5.16, we have  $[M' : N] < \infty$ , which implies  $[P' : N] < \infty$  by the above lemma. By a second application of Lemma 5.16, we deduce that  ${}_N H_P$  is dualizable.

This argument might look circular at first glance, as Lemma 5.16 depends on (5.6). However, Lemma 5.16 only depends on the special case of (5.6) mentioned in footnote 5, and is thus independent of the result of this corollary.  $\square$

Unless the factors are finite-dimensional, the Pimsner–Popa inequality also provides a characterization of the minimal conditional expectation and of the minimal index. For a subfactor  $N \subset M$  of finite index, let  $E_0(m) := [M : N]^{-\frac{1}{2}} R^*(m \otimes 1)R$ , as before.

**Proposition 7.10.** *Assume the factors  $N$  and  $M$  are infinite-dimensional, and  $N \subset M$  is of finite index. In this case,*

*a. if  $0 < \lambda < [M : N]$ , there exists  $x \in M_+$  such that*

$$(7.11) \quad E_0(x) \not\geq \lambda^{-1}x.$$

*In other words,  $\text{Ind}(E_0) \geq [M : N]$ , and therefore, by equation (7.4),  $\text{Ind}(E_0) = [M : N]$ .*

*b. if  $E : M \rightarrow N$  is a conditional expectation and  $E \neq E_0$ , then  $\exists x \in M_+$  such that*

$$(7.12) \quad E(x) \not\geq [M : N]^{-1}x.$$

*In other words,  $\text{Ind}(E) > [M : N]$ .*

*Proof.* a. We let

$$\text{cap} : L^2 N \rightarrow L^2 M \quad \text{and} \quad \text{cup} : L^2 M \rightarrow L^2 M \boxtimes_N L^2 M$$

be normalized duality maps for the bimodule  ${}_N H_M := {}_N L^2 M_M$ . Let  $d = [M : N]^{\frac{1}{2}}$  be the statistical dimension of  $H$ , and let  $e = d^{-1} \text{cup}$  be the Jones projection. Since  $\dim(N) = \infty$ , one can find a right  $M$ -module  $K_M$  such that  $K \boxtimes_M L^2(M)_N$  is isomorphic to  $L^2 N_N$  — use the classification of modules over factors of type different from  $I_n$  to see this. Pick a unitary isomorphism  $u : K \boxtimes_M L^2(M)_N \rightarrow L^2 N_N$  and set  $x := (u \otimes 1)(1 \otimes e)(u^* \otimes 1)$ . We then have

$$E_0(x) = E_0 \left( d^{-1} \text{cap} \left( \text{cup} \left( \text{cap} \left( d^{-1} \text{cup} \right) \right) \right) \right) = d^{-2} \text{cap} \left( \text{cup} \left( \text{cap} \left( d^{-1} \text{cup} \right) \right) \right) = [M : N]^{-1} \text{cap} \left( \text{cup} \left( \text{cap} \left( d^{-1} \text{cup} \right) \right) \right) = [M : N]^{-1},$$

where the dotted line stands for  $K$ . Since  $x$  is a non-zero projection and  $[M : N]^{-1}$  is a scalar, it follows that  $E_0(x) \not\geq \mu x$  for any  $\mu > [M : N]^{-1}$ .

b. We need to check that  $E_0$  minimizes  $\text{Ind}(E)$ . Let  $p_i$  be the minimal central projections of  $N' \cap M = \text{End}({}_N L^2 M_M)$ , let  $d = [M : N]^{\frac{1}{2}}$ , and let  $d_i = [p_i M p_i : p_i N]^{\frac{1}{2}}$ . Note that  $p_i N \subset p_i M p_i$  is an irreducible subfactor, that is  $(p_i N)' \cap p_i M p_i = \mathbb{C}$ . Thus by [16, Proposition 5.3], there exists only one conditional expectation  $p_i M p_i \rightarrow p_i N$ . Using part (a) and Proposition 7.5 we conclude that for  $p_i N \subset p_i M p_i$ , the minimal index coincides with the Longo index. Thus  $d_i = (\text{Ind}(p_i N, p_i M p_i))^{\frac{1}{2}}$ . According to [16, Theorem 5.5], it suffices to check that  $E_0|_{N' \cap M}$  is a trace and that

$$(7.13) \quad E_0(p_i) = \frac{d_i}{\sum d_i}.$$

The first condition was proven in Lemma 4.15. To check the latter, let  $H_i := p_i(L^2(M))$ . We then have

$$\begin{aligned} d_i &= \dim_{(p_i N L^2(p_i M p_i)_{p_i M p_i})} = \dim_{(p_i N p_i(L^2 M)_{p_i M p_i})} \\ &= \dim_{(p_i N p_i L^2 M \boxtimes_M L^2 M p_i)_{p_i M p_i}} = \dim_{(p_i N p_i L^2 M_M)} = \dim(H_i) = \text{[Diagram: A box labeled } p_i \text{ inside a circle, which is inside a larger circle]} \end{aligned}$$

where the second equality is Lemma 2.6, the fourth one holds by equations (5.4) and (5.6), and the last one is given by Lemma 4.20. Note that  $\sum d_i = d$  now follows by equation (5.5). By the definition of  $E_0$ , we therefore have

$$(\sum d_i) \cdot E_0(p_i) = d E_0(p_i) = \text{[Diagram: A box labeled } p_i \text{ inside a circle, which is inside a larger circle]} = d_i. \quad \square$$



**Corollary 7.14.** *Let  $N \subset M$  be infinite-dimensional factors, let  $[M : N]$  be the minimal index, as in Definition 5.10, and let  $\text{Ind}(N, M)$  be the Longo index, as in equation (7.2). Then*

$$(7.15) \quad [M : N] = \text{Ind}(N, M).$$

*Warning 7.16.* As noted in [16], the equality (7.15) fails to be true, for example, for the subfactors  $\mathbb{C} \hookrightarrow M_n(\mathbb{C})$ . The Longo index  $\text{Ind}(N, M)$  is not a good notion of index in the case of finite-dimensional factors.

Now is an appropriate moment to pay our debt to Remark 4.14, by giving a particularly mild condition that ensures that a bimodule is dualizable — compare [17, Theorem 4.1].

**Proposition 7.17.** *Let  ${}_A H_B$  and  ${}_B K_A$  be irreducible bimodules between von Neumann algebras with finite center. If there exist non-zero maps  $\tilde{R} : {}_A L^2(A)_A \rightarrow {}_A H \boxtimes_B K_A$  and  $\tilde{S} : {}_B L^2(B)_B \rightarrow {}_B K \boxtimes_A H_B$ , then  ${}_A H_B$  and  ${}_B K_A$  are dualizable.*

*Proof.* We denote  $\tilde{R}$  by  and  $\tilde{S}$  by . We may assume without loss of generality that  $A$  and  $B$  are factors, and that  $\tilde{R}$  and  $\tilde{S}$  are isometries. Define conditional expectations  $E : B' \rightarrow A$  and  $F : A' \rightarrow B$  by

$$E : x \mapsto \text{[Diagram: A box labeled x inside a circle, which is inside a larger circle]} , \quad F : y \mapsto \text{[Diagram: A box labeled y inside a circle, which is inside a larger circle]} ,$$

where the commutants are taken on  $H$ .

Denote by  $U(A)$  the group of unitary elements of  $A$ . For any non-zero projection  $p \in B'$ , the least upper bound  $\bigvee_{u \in U(A)} upu^*$  belongs to  $A' \cap B' = \text{End}({}_A H_B) = \mathbb{C}$  and is therefore equal to 1. If  $E(p)$  were zero, we would have

$$1 = E(1) = E(\bigvee upu^*) = \bigvee E(upu^*) = \bigvee uE(p)u^* = 0.$$

Thus the conditional expectation  $E$  is faithful, and similarly  $F$  is faithful. It follows from [17, Proposition 4.4] that the inclusion  $A \subset B'$  has finite index. By Lemma 5.16, we then have  $\dim({}_A H_B) = \llbracket B' : A \rrbracket < \infty$ , and so  ${}_A H_B$  is dualizable. The bimodule  ${}_B K_A$  is dualizable for similar reasons.  $\square$

We finish this section by establishing some useful inequalities for the matrix of statistical dimensions  $\llbracket B : A \rrbracket$  — recall Definition 5.13 — associated to a finite homomorphism  $A \rightarrow B$  of von Neumann algebras with finite center. Our proofs are all based on the Pimsner–Popa inequality.

Let  $A_1, B_1 \subset \mathbf{B}(H_1)$  and  $A_2, B_2 \subset \mathbf{B}(H_2)$  be von Neumann algebras such that  $A_i$  commutes with  $B_i$ . The algebras  $A_1 \vee B_1 \subset \mathbf{B}(H_1)$  and  $A_2 \vee B_2 \subset \mathbf{B}(H_2)$  are therefore completions of the corresponding algebraic tensor products  $A_1 \otimes_{alg} B_1$  and  $A_2 \otimes_{alg} B_2$ . Given homomorphisms  $\alpha : A_1 \rightarrow A_2$  and  $\beta : B_1 \rightarrow B_2$ , the induced map  $\alpha \otimes \beta : A_1 \otimes_{alg} B_1 \rightarrow A_2 \otimes_{alg} B_2$  does not always extend to a map  $A_1 \vee B_1 \rightarrow A_2 \vee B_2$ . This will however be the case in the presence of an  $\alpha \otimes \beta$ -equivariant homomorphism  $h : H_1 \rightarrow H_2$ .

**Lemma 7.18.** *Let  $A_i, B_i, H_i$ , and  $h$  be as above. If the algebras  $A_i, B_i$ , and  $A_i \vee B_i$  have finite center and the homomorphisms  $\alpha : A_1 \rightarrow A_2$  and  $\beta : B_1 \rightarrow B_2$  are finite, then the induced map*

$$(7.19) \quad \alpha \otimes \beta : A_1 \vee B_1 \rightarrow A_2 \vee B_2$$

*is a finite homomorphism.*

*Proof.* Let us write  $\vee_{H_1}$  and  $\vee_{H_2}$  for the completions inside  $\mathbf{B}(H_1)$  and  $\mathbf{B}(H_2)$ , respectively. We can then factor the map (7.19) as

$$A_1 \vee_{H_1} B_1 \longrightarrow A_1 \vee_{H_2} B_1 \longrightarrow A_2 \vee_{H_2} B_1 \longrightarrow A_2 \vee_{H_2} B_2.$$

The first map is a projection, and therefore finite. We analyze the second map — the third one is similar. From now on let  $\vee$  mean  $\vee_{H_2}$ . The restriction to  $A'_1 \cap B'_1 = (A_1 \vee B_1)'$  of the minimal conditional expectation  $E_0 : A'_1 \rightarrow A'_2$  satisfies the same Pimsner–Popa bound as  $E_0$ . The homomorphism  $(A_1 \vee B_1)' \rightarrow (A_2 \vee B_1)'$  is therefore finite by Proposition 7.5. Corollary 5.17 finishes the argument.  $\square$

**Proposition 7.20.** *Let  $A$  be an infinite-dimensional factor sitting in a von Neumann algebra  $B$ . If there exists a conditional expectation  $E : B \rightarrow A$  satisfying the Pimsner–Popa bound*

$$(7.21) \quad E(x) \geq \mu^{-1}x \quad \forall x \in B_+,$$

*then  $B$  has finite center. Furthermore, letting  $p_i$  be the minimal central projections of  $B$ , we then have  $\sum [p_i B : A] \leq \mu$ . In other words, we have the inequality*

$$\| [B : A] \| \leq \sqrt{\mu},$$

*where  $\| \cdot \|$  stands for the  $\ell^2$  norm of a vector.*

*Proof.* Let  $q_i \in B$  be non-zero central projections adding up to 1. Since

$$aE(q_i) = E(aq_i) = E(q_i a) = E(q_i)a$$

for all  $a \in A$ , the element  $E(q_i)$  is central in  $A$ , and hence a scalar. From the bound (7.21), we conclude that  $E(q_i) \geq \mu^{-1}$ . Summing up over all indices  $i$ , we deduce

$$1 = E(1) = E\left(\sum q_i\right) = \sum_i E(q_i) \geq \sum_i \mu^{-1},$$

from which it follows that the number of  $q_i$ 's is at most  $\mu$ . The center of  $B$  is therefore finite-dimensional.

Now let  $p_i$  be the minimal central projections of  $B$ , and let  $B_i := p_i B$ . The restriction  $F_i := E|_{B_i} : B_i \rightarrow A$  satisfies all the properties for being a conditional expectation, except that it does not send the unit  $p_i$  of  $B_i$  to 1. The map  $E_i := F_i(p_i)^{-1}F_i$  is therefore a conditional expectation. It satisfies the bound

$$E_i(x) \geq F_i(p_i)^{-1}\mu^{-1}x \quad \forall x \in B_{i+},$$

from which it follows that  $[B_i : A] \leq F_i(p_i)\mu$ . Adding up over indices, we get that  $\sum [B_i : A] \leq \sum F_i(p_i)\mu = E(\sum p_i)\mu = E(1)\mu = \mu$ .  $\square$

The following lemma is, in some sense, dual to Proposition 7.20:

**Proposition 7.22.** *Let  $A = \oplus A_i$  be a sum of finitely many infinite-dimensional factors  $A_i$ , and suppose that  $A$  is a subalgebra of some factor  $B$ . Let  $E : B \rightarrow A$  be a conditional expectation satisfying*

$$(7.23) \quad E(x) \geq \mu^{-1}x \quad \forall x \in B_+.$$

*Letting  $p_i$  be the minimal central projections of  $A$ , we have  $\sum [p_i B p_i : A_i] \leq \mu$ . In other words, we have the inequality*

$$\| [B : A] \| \leq \sqrt{\mu},$$

*where  $\| \cdot \|$  stands for the  $\ell^2$  norm of a vector.*

*Proof.* Under our assumption on  $A$  the optimal  $\mu$  satisfying (7.23) can be identified with the Kosaki index  $\|E^{-1}(1)\|$  of the conditional expectation  $E$ , see [25, Theorem 1.1.6]. By its definition [12, 13], the Kosaki index does not change under tensor product with another factor. In particular, given a type III factor  $R$ , we conclude that the conditional expectation  $E \otimes R : B \otimes R \rightarrow A \otimes R$  satisfies the same bound (7.23) as  $E$ . The index of  $A_i \otimes R$  in  $p_i(B \otimes R)p_i$  being equal to that of  $A_i$  in  $p_i B p_i$ , we may assume without loss of generality that  $B$  is a type III factor.

Let us define  $B_{ij} := p_i B p_j$ . If  $B$  is a type III factor, then the projections  $p_i$  are all Murray-von Neumann equivalent; we can therefore identify each matrix block  $B_{ij}$  with a given algebra, say  $C$ , and get an isomorphism

$$B = \bigoplus_{ij} B_{ij} \cong M_n(C).$$

Taking the composite  $B_{ii} \hookrightarrow B \xrightarrow{E} A \twoheadrightarrow A_i$ , we get a conditional expectation  $E_i : B_{ii} \rightarrow A_i$ . Let  $\lambda_i$  be the smallest number for which the Pimsner–Popa inequality

$$E_i(x) \geq \lambda_i^{-1}x \quad \forall x \in B_{ii+}$$

holds, and note that there exist projections  $e_i \in B_{ii}$  such that  $E_i(e_i) = \lambda_i^{-1}p_i$ ; for example, we can take  $e_i$  to be a Jones projection as in the proof of Proposition 7.10a.

Let  $u_{ij} \in C$  be partial isometries with  $u_{ij}u_{ij}^* = e_i$ ,  $u_{ij}^* = u_{ji}$ , and  $u_{ij}u_{jk} = u_{ik}$ . In particular, we have  $u_{ii} = e_i$ . Consider now the projection  $Q \in M_n(C)$  given by

$$Q_{ij} := \frac{\sqrt{\lambda_i \lambda_j}}{\sum_k \lambda_k} u_{ij}.$$

We then have

$$E(Q) = \bigoplus E_i(Q_{ii}) = \bigoplus E_i\left(\frac{\lambda_i}{\sum_k \lambda_k} e_i\right) = \bigoplus \frac{\lambda_i}{\sum_k \lambda_k} E_i(e_i) = \bigoplus \frac{1}{\sum_k \lambda_k} p_i = \frac{1}{\sum_k \lambda_k}.$$

Combined with the bound (7.23), the above estimate shows that  $\mu \geq \sum \lambda_k$ . To finish the proof, we use the inequality  $\lambda_i \geq [p_i B p_i : p_i A]$ , which follows from (7.11) and (7.12).  $\square$

*Remark 7.24.* We expect that, analogously to Proposition 7.20, when  $A \subset B$  with  $B$  a factor, the existence of a conditional expectation  $B \rightarrow A$  satisfying a Pimsner–Popa bound actually implies that  $A$  has finite-dimensional center.

Given the results of Propositions 7.20 and 7.22 it is natural to ask the following:

**Question 7.25.** Let  $A \subset B$  be von Neumann algebras with finite center, and let  $E : B \rightarrow A$  be a conditional expectation satisfying the Pimsner–Popa bound  $E(x) \geq \mu^{-1}x$ ,  $\forall x \in B_+$ . For which norm  $\| \cdot \|$  on matrices do we then get the inequality  $\| [B : A] \| \leq \sqrt{\mu}$ ?

Finally, we use the previous two propositions to explain the relationship between the minimal index and the operations of relative commutant and of completed tensor product.

**Corollary 7.26.** *Let  $N \subset M \subset A \subset \mathbf{B}(H)$  be subalgebras with  $N$  and  $M$  factors and  $[M : N] < \infty$ . Suppose that one of the two relative commutants  $N' \cap A$  or  $M' \cap A$  is a factor, and that the other one has finite-dimensional center. In this case,*

$$\| [N' \cap A : M' \cap A] \| \leq [M : N].$$

*Proof.* By Corollary 5.17, we know that  $[N' : M'] = [M : N]$ . Let  $E' : N' \rightarrow M'$  be the minimal conditional expectation from  $N'$  to  $M'$ . If  $a \in A' \subset N'$  and  $x \in N' \cap A$ , then we have  $aE'(x) = E'(ax) = E'(xa) = E'(x)a$ , showing that  $E'(x) \in M' \cap A$ . The restriction  $E := E'|_{N' \cap A}$  is therefore a conditional expectation from  $N' \cap A$  to  $M' \cap A$ . By the Pimsner–Popa inequality for  $E'$ , we know that

$$E(x) \geq [N' : M']^{-1}x = [M : N]^{-1}x, \quad \forall x \in N' \cap A.$$

By Proposition 7.20 or Proposition 7.22, it follows that  $\| [N' \cap A : M' \cap A] \| \leq [M : N]$ .  $\square$

**Corollary 7.27.** *Let  $N \subset M \subset \mathbf{B}(H)$  be factors with  $[M : N] < \infty$ , and let  $A \subset \mathbf{B}(H)$  be an algebra that commutes with  $M$ . Suppose that one of the algebras  $N \vee A$  and  $M \vee A$  is a factor, and that the other one has finite-dimensional center. In this case,*

$$\| [M \vee A : N \vee A] \| \leq [M : N].$$

*Proof.* By the previous corollary, we have  $\| [N' \cap A' : M' \cap A'] \| \leq [M : N]$ . The result now follows from Corollary 5.17, because  $(M \vee A)' = M' \cap A'$  and  $(N \vee A)' = N' \cap A'$ .  $\square$

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